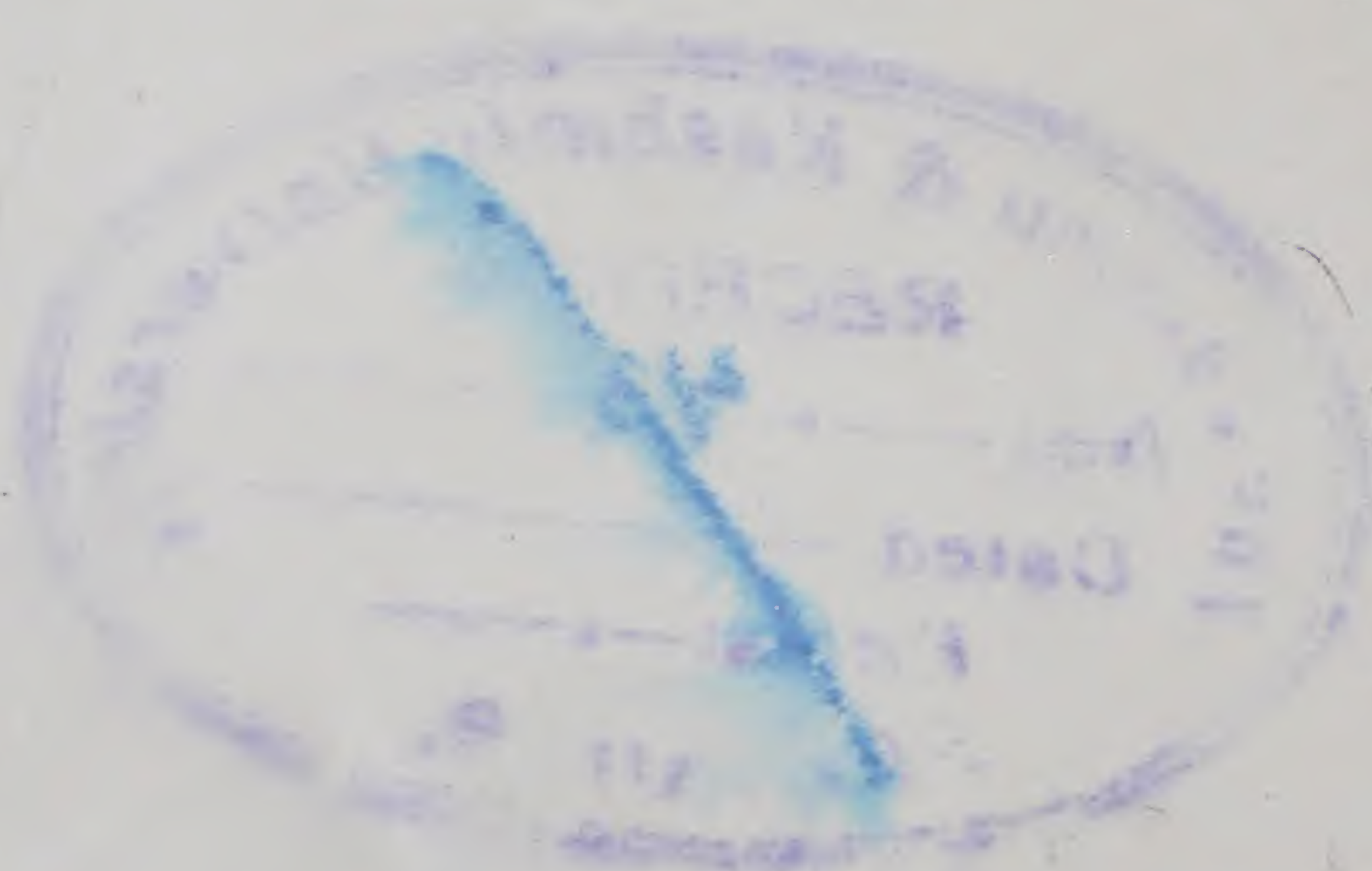




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PREFACE

THE book completely covers the whole course in Algebra for the Matriculation examination of the Indian Universities. It has been divided into two parts: Part I for pre-matriculation, and Part II for the matriculation class. Each part is divided into three sections, each of which covers one term's work. At the end of each section Sectional Revision in the form of Test Papers has been added.

The elementary portions have been treated in the first three sections and the more advanced in the last three.

The main features of the book are:

(i) The subject is introduced in the form of *Literal Arithmetic*, with special emphasis on *generalisation*.

(ii) A separate and comprehensive chapter on *Directed Numbers* forms the basis of Algebra.

(iii) *Formulae* and *Factors* have been exhaustively treated at various stages and their application in other rules is also worked out in a separate chapter.

(iv) *Equations* have been properly emphasised and they are treated at various stages, as (i) simple equations, (ii) fractional equations, (iii) simultaneous equations, (iv) exponential equations, (v) equations involving surds, and (vi) quadratic equations.

(v) The application of the method of *cross-multiplication* is fully worked out in (a) simultaneous equations, (b) elimination and (c) in finding the necessary relations to satisfy a given condition.

(vi) The fractional and negative *Indices* have been properly introduced and due attention has been paid to *Surds*.

(vii) An exhaustive chapter on *Elimination* has been added.

(viii) *Conditional Identities* with their application have been included.

(ix) In order to apply the principles of proportion to other rules, *Ratio* and *Proportion* have been treated quite early in Part II.

(x) There is a separate chapter on *linear, statistical* and *quadratic* graphs but graphic illustrations have been employed wherever necessary and useful.

(xi) One chapter on *Homogeneity, Symmetry, Cyclic Order* and *Indeterminate Co-efficients*, another on the *Remainder Theorem* with its application in factors and a third on *Conditional Identities, Harder Factors, Σ -notation, Fractions with denominators in cyclic order, etc.*, have been added for those students who possess special aptitude for Mathematics.

(xii) 100 selected questions covering the whole course have been given at the end as miscellaneous exercises.

*—marked articles, examples and exercises are meant for brighter students only and not for the whole class.

In preparing the Fifth Edition of this book an opportunity has been taken to thoroughly revise it, with a view to increase its practical utility as a text-book. It is hoped that this revised edition will meet with the same good appreciation from all its users as was accorded to the previous four editions.

Late Professor of Mathematics,
Central Training College, Lahore.

G. M. Vohra,
P.E.S. (Retd.)

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PART I

CHAPTER I

LITERAL ARITHMETIC

1. **Signs and symbols.** In Arithmetic figures 0, 1, 2, 3, 9 are used to represent numbers and these figures can have only *one value*, whereas in Literal Arithmetic, in addition to figures, letters such as a, b, c, \dots, x, y, z are used, to which *any value* may be assigned. For instance, when we speak of 12 pencils, we have a particular number of pencils in our mind, but when we speak of x pencils, we may be thinking of 5 pencils, 7 pencils, or in fact *any number* of pencils.

NOTE. One particular letter has only one particular value in the same problem.

2 In Arithmetic as well as in Literal Arithmetic, the signs $+, -, \times, \div, =, >, <, \therefore$ and \therefore have got the same meaning.

In Literal Arithmetic we use \sim as the *sign of difference*, as distinguished from the sign of subtraction; for example $a \sim b$ means the difference between a and b , where a may or may not be greater than b .

In Arithmetic 3×4 is sometimes written as 3.4, so in Literal Arithmetic $a \times b$ is written as $a.b$; but in Literal Arithmetic this dot is generally omitted and $a \times b$ is written as ab . Similarly, $5 \times a \times b$ is written as $5ab$.

3 In Arithmetic the sign of multiplication cannot be omitted without changing the meaning; for example $3 \times 4 = 12$ or $10 + 2$ and $34 = 30 + 4$.

NOTE. The dot used as the sign of multiplication is written near the bottom, whereas the dot used as a decimal point is written near the top, as $3.4 = 3 \times 4$ and $3\cdot4 = 3 + \frac{4}{10}$.

Just as in Arithmetic $3 + 3 + 3 + 3 + 3$ is 5 times 3 or 5.3, so in Literal Arithmetic $a + a + a + a + a$ is 5 times a or 5. a or $5a$. Similarly, $ab + ab + ab + ab = 4.ab$ or $4ab$.

Definitions. In $5a$ and $4ab$ the numbers 5 and 4 are called the numerical co-efficients of a and ab respectively.

NOTE. (i) When the numerical co-efficient is 1, it is not expressed; thus in ab which is really $1 \times ab$, the numerical co-efficient is 1.

(ii) 0×4 means 4 times 0 and is equal to 0; similarly, $0 \times a$ means a times 0 and is equal to 0.

Again, 4×0 means 0 times 4 which is clearly equal to 0; similarly $a \times 0$ means 0 times a and is equal to 0.

EXERCISE 1. (Oral)

1. Find the total number of runs scored by Rashid, Hamid and Saeed :—

- (i) If Rashid scores 12, Hamid 7 and Saeed r .
- (ii) „ „ 12 „ q „ „ r .
- (iii) „ „ p „ q „ „ r .

2. What is the total length of three lines L , M , and N :

- (i) If L is $3''$, M is $9''$ and N is z'' ?
- (ii) „ L „ $3''$, M „ y'' and N „ z'' ?
- (iii) „ L „ x'' , M „ y'' and N „ z'' ?

3. Write in symbols :

- (i) 5 added to x .
- (ii) x increased by 6.
- (iii) x increased by y .
- (iv) 5 added to x , and y added to the result.

4. Express in words :

- (i) $x + 7$.
- (ii) $p + q$.
- (iii) $x + 11 + y$.
- (iv) $a + b + c$.

5. Find the value of $x + y + z$:

- (i) When $x = 3$, $y = 2$, $z = 4$.
- (ii) When $x = \frac{2}{3}$, $y = \frac{3}{4}$, $z = \frac{1}{6}$.
- (iii) When $x = \cdot 24$, $y = \cdot 16$, $z = \cdot 15$.
- (iv) When $x = 0$, $y = 5$, $z = 13$.

6. How many rupees are left with me :

- (i) If I spend Rs. 7 out of Rs. x ?
- (ii) „ „ Rs. y „ „ Rs. x ?
- (iii) „ „ Rs. y and then Rs. z out of Rs. x ?

7. Write in symbols :

- (i) 9 subtracted from 15. (ii) 9 subtracted from m .
- (iii) 11 diminished by x . (iv) a taken from b .
- (v) The difference between m and n .
- (vi) The sum of 11 and b diminished by c .
- (vii) y subtracted from x and z added to the result.

8. Express in words :

- (i) $p - 7$. (ii) $m \sim n$.
- (iii) $l + m - n$. (iv) $p - q + r$.
- (v) $x - y - z$. (vi) $a + b - c - d$.

9. Find the value of $a + b - c - d$:

- (i) When $a = 2$, $b = 7$, $c = 1$, $d = 3$.
- (ii) When $a = \frac{1}{2}$, $b = \frac{3}{4}$, $c = \frac{1}{4}$, $d = \frac{1}{6}$.
- (iii) When $a = \cdot 15$, $b = \cdot 8$, $c = \cdot 02$, $d = \cdot 46$.

10. Find the area of a rectangle :

- (i) If its length is 14 ft. and breadth is 12 ft.
- (ii) „ „ 14 ft. „ „ b ft.
- (iii) „ „ l ft. „ „ b ft.

11. (i) How many feet are there in 6 yds. ?

- (ii) „ „ „ „ „ x yds. ?
- (iii) „ „ inches „ „ m ft. ?
- (iv) „ „ annas „ „ p rupees ?
- (v) „ „ centimetres „ „ n metres ?

12. If 1 horse costs Rs. r , what will n horses cost ?

13. If a person travels at the rate of m miles an hour, how far will he go in t hours ?

14. Write in symbols :

- (i) 5 multiplied by x . (ii) m multiplied by 5.
 (iii) The product of m and n .
 (iv) 7 multiplied by a and the product by b .

15. Write down *briefly* the following and state the numerical co-efficient in each case :

- (i) $x + x + x + x$. (ii) $a + a + a + a + a$.
 (iii) $pq + pq + pq$. (iv) $abc + abc + abc + abc$.

16. Express in words :

- (i) $8 \times x$. (ii) xy . (iii) $3xy$.
 (iv) $5xyz$. (v) $7abc$.

17. Distinguish between :

- (i) 39, 3.9, 3·9. (ii) 47, 4×7 , 4.7, 4·7. (iii) mn , $m \times n$, $m.n$.

18. Find the value of $5mn + 3pq$:

- (i) When $m = 6$, $n = 2$, $p = 4$, $q = 1$.
 (ii) When $m = \cdot 7$, $n = \cdot 4$, $p = \cdot 5$, $q = \cdot 24$.
 (iii) When $m = \frac{1}{2}$, $n = \frac{7}{10}$, $p = \frac{5}{8}$, $q = \frac{2}{3}$.
 (iv) When $m = 7$, $n = 1$, $p = 0$, $q = 2$.

19. Write in symbols :

- (i) 15 divided by 7. (ii) 15 divided by x .
 (iii) m divided by 7. (iv) p divided by q .
 (v) The sum of p and q divided by m .
 (vi) The difference between a and b divided by c .

20. Express in words :

- (i) $\frac{x}{y}$. (ii) $p \times r \div s \times q$. (iii) $\frac{5mn}{2n}$.
 (iv) $\frac{3p+q}{5}$. (v) $\frac{7p+2q}{p-q}$.

21. (i) How many yards are there in $18x$ inches ?

(ii) „ feet „ in $30ab$ inches ?

(iii) „ rupees „ in $48m$ annas ?

(iv) „ metres „ in $250x$ centimetres ?

22. If the price of 7 chairs is Rs. 28*m*, what is the price of 1 chair? The price of *x* chairs?

23. Find the value of $\frac{3ab+4cd}{2ad}$:

- (i) If $a = 2$, $b = 5$, $c = 1$, $d = 4$.
 (ii) If $a = .4$, $b = .5$, $c = .16$, $d = 2.5$.
 (iii) If $a = \frac{1}{3}$, $b = \frac{3}{4}$, $c = \frac{2}{3}$, $d = \frac{1}{2}$.

24. Write in symbols :

- (i) Five added to 6 is equal to 11. $[6 + 5 = 11]$
 (ii) Four and five make nine.
 (iii) Two and two make four.
 (iv) Four times three equals three times four.
 (v) Twenty divided by five is equal to eighty divided by twenty.
 (vi) Half of p is equal to q .
 (vii) Since five times x is equal to forty, therefore x is equal to eight. $[\because 5x = 40, \therefore x = 8.]$
 (viii) Since one-third of y is equal to fifteen, therefore y is equal to forty-five.
 (ix) Since seven times m is greater than thirty-five, therefore m is greater than five.
 (x) Since six times p is less than eighteen, therefore p is less than three.

25. Express in words :

- (i) $x + 7 = 16$.
 (ii) $x - 11 = 5$.
 (iii) $5x = 30$.
 (iv) $\frac{2x}{9} = 6$.
 (v) $2a + 3b = 5$.
 (vi) $\frac{2a + 3b}{7} = 10$.
 (vii) $\frac{3a - 2b}{5} = 12$.
 (viii) $\because 5x - 3 = 12, \therefore x = 3$.
 (ix) $\because 7x - 2 > 12, \therefore x > 2$.
 (x) $\because 2x + 5 < 15, \therefore x < 5$.

2. Definitions and Fundamental Laws. Just as in Arithmetic

5×5 is written briefly 5^2

$5 \times 5 \times 5$ „ „ „ 5^3

$5 \times 5 \times 5 \times 5$ „ „ „ 5^4 , and so on ;

similarly, in Literal Arithmetic

$a \times a$ is written briefly a^2 { Read as '*a squared*' or '*the square of a*'.

$a \times a \times a$ „ „ „ a^3 { Read as '*a cubed*' or '*the cube of a*'.

$a \times a \times a \times a$ „ „ „ a^4 { Read as '*a to the fourth*' or '*the fourth power of a*'.

If in a product the same factor is repeated a certain number of times, as above, the *product* is called the **power** of that factor and the small figure placed above the factor to indicate *the number of times* it is repeated as a factor in that product is called its **index**.

Thus in 7^2 , 6^4 , a^5 , b^3 the indices are, 2, 4, 5, 3 respectively.

An **expression** is a collection of symbols—that is, of letters, figures and signs. Thus $3a$, $7a^2b$, $4a + 5b$, $3a^2 + 2ab - b^2$ are all expressions.

The parts of an expression connected by the sign + or — are called the **terms** of the expression.

$3a$ is an expression of *one term* or a **monomial**.

$4a + 5b$ is an expression of *two terms* or a **binomial**.

$3a^2 + 2ab - b^2$ is an expression of *three terms* or a **trinomial**.

Terms having the same letters are called **like terms** and terms having different letters are called **unlike terms**, e.g., $3abc$ and $7abc$ are like terms, and $3abc$ and $7bcd$ are unlike terms.

An expression containing only one term is also called a **Simple Expression**.

An expression containing more than one term is called a **Compound Expression**.

Example 1. Distinguish between $3x$ and x^3 .

$$3x = x + x + x \text{ and } x^3 = x \times x \times x.$$

Example 2. If $a=4$, $b=3$ and $c=2$, find the value of:

(i) $5ab$. (ii) a^2b^3 . (iii) $3a^b$. (iv) $4a^3 - 3b + 5c$.

(i) $5ab = 5 \times a \times b = 5 \times 4 \times 3 = 60$.

(ii) $a^2b^3 = a \times a \times b \times b \times b = 4 \times 4 \times 3 \times 3 \times 3 = 432$.

(iii) $3a^b = 3 \times a^3 = 3 \times a \times a \times a = 3 \times 4 \times 4 \times 4 = 192$.

(iv) $4a^3 - 3b + 5c = 4 \times a \times a \times a - 3 \times b + 5^2$
 $= 4 \times 4 \times 4 \times 4 - 3 \times 3 + 5 \times 5$
 $= 256 - 9 + 25 = 272$.

EXERCISE 2.

1. Write briefly :

(i) $7 \times 7 \times 7 \times 7$.

(ii) $a \times a \times a \times a \times a$.

(iii) $a \times p \times p \times p$.

(iv) $1 \times a \times b \times b$.

(v) $\cdot 01 \times \cdot 01 \times \cdot 01$.

(vi) $\frac{1}{a} \times \frac{1}{a} \times \frac{1}{a} \times \frac{1}{a}$

(vii) $3 \times m \times m + 4 \times n \times n$. (viii) $a \times a \times n \times n \times n$.

(ix) $pppp + ppp$.

(x) $a \times x \times x \times x + b \times y \times y \times y$.

2. Read the following :

(i) p^3 . (ii) x^4 . (iii) r^6 . (iv) n^{15} .

3. State the value of :

(i) 5^4 . (ii) 7^2 . (iii) 2^5 . (iv) 3^4 .

4. Distinguish between :

(i) $5a$ and a^5 . (ii) p^4 and $4p$.

(iii) Three times 5 and 5 cubed.

5. If $a=3$, $b=5$, $c=2$, $x=1$, $y=4$, find the value of :

(i) $2a^3 + 3b^2$.

(ii) $5a^4 - 2b^2 + 4x^3$.

(iii) $11a^3 + 6b^3 + 2x^c + y^3$. (iv) $2ab^3 + 3a^2c^2 + 4xy^3$.

(v) $a^c + bx$.

(vi) $ya + x^b + cx$.

6. Find the value of $x^3 + x^2y + xy^2 + y^3$ from the values of x and y given in the table :

x	3	2	4	1	4
y	1	3	2	3	1

Just as in Arithmetic, we have

$$(i) \quad 4 \times 3 = 3 \times 4,$$

$$(ii) \quad 5 \times 4 \times 3 = 5 \times 3 \times 4 = 4 \times 5 \times 3 = 4 \times 3 \times 5 \\ = 3 \times 5 \times 4 = 3 \times 4 \times 5, \text{ and so on;}$$

similarly, in Literal Arithmetic, we have

$$(i) \quad a \times b = b \times a,$$

$$(ii) \quad a \times b \times c = a \times c \times b = b \times c \times a = b \times a \times c \\ = c \times a \times b = c \times b \times a,$$

$$\text{or} \quad abc = acb = bca = bac = cab = cba.$$

Example 3. Simplify $3a \times 5b$.

$$3a \times 5b = 3 \times a \times 5 \times b \\ = 3 \times 5 \times a \times b \\ = 15ab.$$

First Law of Indices.

$$(i) \quad \because a^3 = a \times a \times a \text{ and } a^2 = a \times a,$$

$$\therefore a^3 \times a^2 = a \times a \times a \times a \times a \\ = a^5 = a^{3+2}.$$

$$(ii) \quad \because a^3 = a \times a \times a \text{ and } a^4 = a \times a \times a \times a$$

$$\therefore a^3 \times a^4 = a \times a \times a \times a \times a \times a \times a \\ = a^7 = a^{3+4}.$$

Similarly, $a^4 \times a^5 = a^{4+5}$

and $a^5 \times a^6 = a^{5+6}$

or in general

$$a^m \times a^n = a^{m+n},$$

where m and n are ordinary arithmetical integers.

Law. The index of the product of two factors which are powers of the same quantity is the sum of their indices.

Example 4. Multiply $5a^3b^2 \times 3a^2b^4$.

$$\begin{aligned} 5a^3b^2 \times 3a^2b^4 &= 5 \times a^3 \times b^2 \times 3 \times a^2 \times b^4 \\ &= 5 \times 3 \times a^3 \times a^2 \times b^2 \times b^4 \\ &= 15a^{3+2}b^{2+4} \\ &= 15a^5b^6. \end{aligned}$$

7. Prove that :

(i) $3^4 \times 3^2 = 3^6$.

(ii) $6^4 \times 6^7 = 6^{11}$.

8. Write down the values of the following :

(i) $5^3 \times 5^2 \times 5^2$.

(ii) $7^2 \times 7^3 \times 7$.

(iii) $a^4 \times a^2 \times a^7$.

(iv) $x^6 \times x^3 \times x^4$.

9. Multiply :

(i) $2ab$ by $3cd$.

(ii) $6x^3y$ by $2xy^2$.

(iii) a^2bc by ab^2c .

(iv) $a^2b \times ab \times ac^2$.

(v) $3a^2b^3c^2$ by $5a^3b^4c$.

(vi) $2p^3q \times pq^2 \times 3pq$.

(vii) $4m^2n \times 3mn^3 \times 5mnp^3 \times n^3p^2$.

Just as in Arithmetic, we have

$$\frac{12}{4} = \frac{4 \times 3}{4} = 3$$

so in Literal Arithmetic, we have

$$\frac{ab}{a} = \frac{a \times b}{a} = b.$$

Example 5. Simplify $\frac{12xy}{4y}$.

$$\frac{12xy}{4y} = \frac{12 \times x \times y}{4 \times y} = 3x.$$

Second Law of Indices.

$$(i) \ a^5 \div a^3 = \frac{a^5}{a^3} = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a^2 = a^{5-3}.$$

$$(ii) \ a^7 \div a^4 = \frac{a^7}{a^4} = \frac{a \times a \times a \times a \times a \times a \times a}{a \times a \times a \times a} = a^3 = a^{7-4}.$$

Similarly, $a^9 \div a^5 = a^{9-5}$ and $a^{11} \div a^6 = a^{11-6}$,

or in general $a^m \div a^n = a^{m-n}$,

where m and n are ordinary arithmetical integers and $m > n$.

Law. When a power of a quantity is divided by a lower power of the same quantity, the index of the quotient is equal to the index of the dividend diminished by the index of the divisor.

Example 6. Simplify $\frac{18x^5y^3}{6x^4y}$.

$$\begin{aligned} \frac{18x^5y^3}{6x^4y} &= \frac{18 \times x^5 \times y^3}{6 \times x^4 \times y} \\ &= 3x^{5-4}y^{3-1} \\ &= 3xy^2. \end{aligned}$$

10. Prove that:

$$(i) \ 5^7 \div 5^3 = 5^{7-3}.$$

$$(ii) \ 7^{12} \div 7^4 = 7^{12-4}.$$

11. Write down the values of:

$$(i) \ a^{17} \div a^{12}.$$

$$(ii) \ p^{24} \div p^{15}.$$

$$(iii) \ m^{13} \div m^7.$$

$$(iv) \ x^{24} \div x^{12}.$$

12. Divide:

$$(i) \ 12x^4y^6 \text{ by } 6x^2y^2.$$

$$(ii) \ 10m^7n^5 \text{ by } 2m^4n^4.$$

$$(iii) \ 8p^5q^4r^3 \text{ by } 4p^3q^2r^3.$$

$$(iv) \ 4m^6n^3p^5 \text{ by } 2m^2n^3p^4.$$

$$(v) \ 15l^6m^2n^9 \text{ by } 5l^4m^2n^2.$$

$$(vi) \ 24x^5y^6z^3w^4 \text{ by } 4w^2x^2y^2z.$$

Third Law of Indices.

$$(i) \ (a^3)^2 = a^3 \times a^3 = a^6 = a^{3 \times 2}.$$

$$(ii) \ (a^2)^3 = a^2 \times a^2 \times a^2 = a^6 = a^{2 \times 3}.$$

$$(iii) \ (a^2)^5 = a^2 \times a^2 \times a^2 \times a^2 \times a^2 = a^{10} = a^{2 \times 5}.$$

From these and similar examples it follows that

$$(a^m)^r = a^{mr},$$

where m and r are ordinary arithmetical integers.

Law. When a quantity, raised to a power, is again raised to a certain power, the index of the result is the product of the indices.

13. Prove that :

$$(i) (7^2)^5 = 7^{10}.$$

$$(ii) (8^3)^4 = 8^{12}.$$

$$(iii) (a^3)^3 = a^9.$$

$$(iv) (x^4)^4 = x^{16}.$$

14. Write down the values of the following :

$$(i) (2^3)^2.$$

$$(ii) (3^2)^3.$$

$$(iii) (a^m)^3.$$

$$(iv) (x^3)^m.$$

$$(v) (p^4)^n.$$

$$(vi) (y^3)^5.$$

\sqrt{a} or $\sqrt[2]{a}$ is read as the square root of a .

$\sqrt[3]{a}$ is read as the cube root of a .

$\sqrt[4]{a}$ is read as the fourth root of a .

Example 7. Prove that:

$$(i) \sqrt{a^{12}} = a^6.$$

$$(ii) \sqrt[3]{a^{12}} = a^4.$$

$$(iii) \sqrt[4]{a^{12}} = a^3.$$

$$(i) \text{ Since } a^{12} = a^6 \times a^6 = (a^6)^2,$$

$$\therefore \sqrt{a^{12}} = \sqrt{(a^6)^2} = a^6.$$

$$(ii) \text{ Since } a^{12} = a^4 \times a^4 \times a^4 = (a^4)^3,$$

$$\therefore \sqrt[3]{a^{12}} = \sqrt[3]{(a^4)^3} = a^4.$$

$$(iii) \text{ Since } a^{12} = a^3 \times a^3 \times a^3 \times a^3 = (a^3)^4,$$

$$\therefore \sqrt[4]{a^{12}} = \sqrt[4]{(a^3)^4} = a^3.$$

15. Prove that :

$$(i) \sqrt{10^4} = 10^2.$$

$$(ii) \sqrt{x^{10}} = x^5.$$

$$(iii) \sqrt{9a^2b^2} = 3ab.$$

$$(iv) \sqrt{36^3} = 6^3.$$

$$(v) \sqrt[3]{a^6} = a^2.$$

$$(vi) \sqrt[3]{8x^3y^3} = 2xy.$$

$$(vii) \sqrt[4]{16} = 2.$$

$$(viii) \sqrt[4]{81} = 3.$$

16. Write down the square root of :

$$(i) 5^4.$$

$$(ii) 4^6.$$

$$(iii) x^{12}.$$

$$(iv) 4x^6.$$

$$(v) 9y^8.$$

$$(vi) 25a^2b^4.$$

17. Write down the cube root of :

$$(i) 64a^9.$$

$$(ii) 216x^{12}.$$

$$(iii) 125y^{27}.$$

18. Write down the fourth root of :

$$(i) 16a^8.$$

$$(ii) 81b^{12}.$$

$$(iii) 256c^{20}.$$

19. If $y = 3x^2 + 5x$ and $x = 2$, find the value of y .

20. If x is equal to the square of y , $y = 3z$ and $z = 2$, what is the value of xyz ?

3. Addition and Subtraction. Just as in Arithmetic,

$$8 \times 15 + 5 \times 15 + 7 \times 15 = (8 + 5 + 7) \times 15 \text{ or } 20 \times 15 \text{ and}$$

$$8 \times 15 - 5 \times 15 = (8 - 5) \times 15 = 3 \times 15,$$

similarly, in Literal Arithmetic,

$$8 \times a + 5 \times a + 7 \times a = (8 + 5 + 7) \times a \text{ or } 20a$$

$$\text{and } 8 \times a - 5 \times a = (8 - 5) \times a \text{ or } 3a.$$

EXERCISE 3.

Write down the value of :

$$1. 4a^2 + a^2 + 6a^2.$$

$$2. 3ab + 4ab + 2ab.$$

$$3. p + p + p + p + \dots \text{to 15 terms.}$$

$$4. 9xy + 7xy + 12xy.$$

$$5. 13ab - 7ab.$$

$$6. 17abc - 12abc.$$

$$7. 9a^2 - 4a^2.$$

$$8. 15x^2 - 11x^2.$$

$$9. (i) 7a + 5a.$$

$$10. (i) 8x - 2x.$$

$$(ii) 7 \times 13 + 5 \times 13.$$

$$(ii) 8 \times 5 - 2 \times 5.$$

$$11. (i) 13x + x.$$

$$12. (i) 15y - y.$$

$$(ii) 13 \times 4 + 4.$$

$$(ii) 15 \times 7 - 7.$$

Example 1. Simplify $3a + 4a + b + 5b$.

Since $3a + 4a = 7a$ and $b + 5b = 6b$,

$$\therefore 3a + 4a + b + 5b = 7a + 6b.$$

Example 2. Simplify $5x - 2x + 3y - y + 1$.

Taking the like terms, we have

$$5x - 2x = 3x \text{ and } 3y - y = 2y,$$

$$\therefore 5x - 2x + 3y - y + 1 = 3x + 2y + 1$$

Example 3. Simplify $4xy + 3x + 2y - x - y$.

Taking the like terms together, we have

$$4xy + 3x + 2y - x - y = 4xy + \underline{3x - x} + \underline{2y - y} = 4xy + 2x + y.$$

Example 4. Simplify $5a^2 + a + 7 + 3a - 2a^2 - 6$.

Taking the like terms together, we have

$$\begin{aligned} 5a^2 + a + 7 + 3a - 2a^2 - 6 &= \underline{5a^2 - 2a^2} + \underline{a + 3a} + \underline{7 - 6} \\ &= 3a^2 + 4a + 1. \end{aligned}$$

Simplify :

- | | |
|-------------------------------------|--|
| 13. $a + a + b + b$. | 14. $x + x + y + y + y$. |
| 15. $a + b + a + b$. | 16. $x + y + x + y + x$. |
| 17. $x + 5 + x + 2 + x$. | 18. $x + y + 3 + x + 1$. |
| 19. $a + a + b - a$. | 20. $5x + 7y - 2x + 4$. |
| 21. $a + b - c + a - b + c$. | 22. $5a + b - 2c + b - 2a$. |
| 23. $4xy + 3xz - 2xy$. | 24. $a^2 + 5a^2 - 3a$. |
| 25. $x^3 + 2x^2 + 4x - x^2$. | 26. $p^2 - 3p + 2 + p^2 + 4p + 1$. |
| 27. $a^4 + 11a^2 + 1 - 7a^2$. | 28. $9 - a^2 + 4a - a^2 - 3a + 1$. |
| 29. $a^3 + a + a^2 + 1 + a + a^2$. | 30. $7x^3 + 2x^2 + 2x + 4 - 5x^2 - 4x$. |

An expression such as $3x^5 + 2x + 7x^3 + x^2 + 5x^4 + 1$ cannot be simplified, for all its terms are unlike, but it is useful to arrange its terms, either (i) in *descending powers* of x , i.e., beginning with the highest power of x , then the next highest, and so on; for example, $3x^5 + 5x^4 + 7x^3 + x^2 + 2x + 1$, or (ii) in *ascending powers* of x , i.e., beginning with the term independent of x , then the x term, then the x^2 term, and so on; for example, $1 + 2x + x^2 + 7x^3 + 5x^4 + 3x^5$.

31. Arrange $4 + 5x^2 + x^3 + 7x^4 + 3x$ in (i) descending powers of x , (ii) ascending powers of x .

32. Arrange $a^2 + a^7 + 5a^3 + 1 + 4a^6$ in (i) descending powers of a , (ii) ascending powers of a .

33. Simplify $x^4 + 3x^2 + 4x + 5 - x + 2x^3 - x^2$ and arrange the answer in ascending powers of x .

34. Simplify $8 + x^2 + 3x^4 + 2x^2 + 7x - 4x^3 + 5x^2 - 3x + 1$ and arrange the answer in descending powers of x .

35. The number 435 can be written $4 \cdot 10^2 + 3 \cdot 10 + 5$. Write similarly 374 and in the answer substitute x for 10 and x^2 for 10^2 or 100.

Can you simplify $3x^2 + 7x + 4$?

36. The number 3048 can be written $3 \cdot 10^3 + 4 \cdot 10 + 8$. Write similarly 5704 and in the answer substitute x^2 for 10^2 and x^3 for 10^3 .

Can you simplify $5x^3 + 7x^2 + 4$?

4. Highest Common Factor, Lowest Common Multiple, Fractions.

Of the two expressions $6a^3b^2$, $8a^2b^3$, there are several common factors, viz., 2, $2a$, $2a^2$, $2b$, $2b^2$, $2ab$ and $2a^2b^2$. But $2a^2b^2$ is the highest of all these factors in power and contains in it every other common factor.

$2a^2b^2$ is called the Highest Common Factor (H. C. F.) of $6a^3b^2$ and $8a^2b^3$.

The Lowest Common Multiple (L. C. M.) of $6a^3b^2$ and $8a^2b^3$ is the lowest expression which is exactly divisible by $6a^3b^2$ as well as $8a^2b^3$; therefore it must contain the L.C.M. of 6 and 8, i.e., 24 as a factor and also it must be divisible by a^3b^2 and a^2b^3 , i.e., it must contain a^3b^3 as a factor, for it is the lowest in power of all the common multiples like a^3b^4 , a^4b^3 , a^4b^4 &c. Hence $24a^3b^3$ is the L.C.M. of the two given expressions.

Example 1. Find (i) the H.C.F. and (ii) the L.C.M. of $9x^2y^3$ and $12xy^2z$.

(i) 3 is the H.C.F. of the co-efficients; of the powers of x , i.e., x^2 and x , we take the common factor, x ; of the powers of y , i.e., y^3 and y^2 , we take the common factor, y^2 ; we neglect z , as it is not a factor of $9x^2y^3$.

Multiplying the common factors we have taken, i.e., 3, x and y^2 , we get the H.C.F. $3xy^2$.

(ii) 36 is the L.C.M. of 9 and 12; of the powers of x we take the highest x^2 ; of the powers of y we take the highest y^3 and of the powers of z , the highest z . Multiplying these together we get the L.C.M. $36x^2y^3z$.

EXERCISE 4.

1. Of the expressions $3a$, $6ab$, $4a^2$, $2ab$, $12a^3b^3$, $12a^2b^2$, $6a^2b$, $16a^4$, ab , $3a^2b$, $2ac$, $6a^2c$, $6abc^2$, which are factors of: (i) $6a^2$, (ii) factors of $3ab$, (iii) common factors of $6ab$ and $4a^2$, (iv) multiples of $4a^2$, (v) multiples of $3ab$ (vi) common multiples of $2a^2$ and $3ab$? (Oral).

2. Find the H.C.F. and the L.C.M. of:

- (i) $4a^2b^2$, $6a^3b^3$. (ii) $12a^3b$, $18ab^3$.
 (iii) $12ab$, $3a^2b$. (iv) $39a^3b^4$, $52a^2b^5$.
 (v) $4a^2b$, $6a^3b^3$, $3abr^2$. (vi) $6ab^2$, $8a^2b^2$, $12a^2bc$, $18abc^2$,

3. Find by factors the H.C.F. and the L.C.M. of:

- (i) $2^3.7^2$, $7^3.2$. (ii) $3^2.5^3$, $3^3.5^2$.
 (iii) 864, 720. (iv) 1050, 840.

Example 2. Simplify $\frac{6x^3y^2}{4xy^3z}$.

The H.C.F. of the numerator and the denominator is $2xy^2$. Divide the numerator and the denominator by $2xy^2$.

$$\therefore \frac{6x^3y^2}{4xy^3z} = \frac{3x^2}{2yz}.$$

Example 3. Express $\frac{4x}{3y}$ as an equivalent fraction with denominator $6ay$.

If $6ay$ be divided by $3y$, the quotient is $2a$.

Multiplying the numerator and denominator by $2a$, we have

$$\frac{4x}{3y} = \frac{2a \times 4x}{2a \times 3y} = \frac{8ax}{6ay}.$$

Example 4. Express $\frac{2a^2x^2}{3b^2y^2}$ as an equivalent fraction with numerator $12a^2x^2z^2$.

If $12a^2x^2z^2$ be divided by $2a^2x^2$, the quotient is $6z^2$.
Multiplying the numerator and the denominator by $6z^2$, we have

$$\frac{2a^2x^2}{3b^2y^2} = \frac{6z^2 \times 2a^2x^2}{6z^2 \times 3b^2y^2} = \frac{12a^2x^2z^2}{18b^2y^2z^2}.$$

Example 5. Reduce $\frac{5}{3x}$ and $\frac{7}{4x}$ to fractions having the least common denominator.

The L.C.M. of $3x$ and $4x = 12x$.

$$\therefore \frac{5}{3x} = \frac{4 \times 5}{4 \times 3x} = \frac{20}{12x} \text{ and } \frac{7}{4x} = \frac{3 \times 7}{3 \times 4x} = \frac{21}{12x}.$$

Example 6. (i) Add $\frac{3}{4x}$ and $\frac{1}{10x}$.

(ii) Subtract $\frac{1}{10x}$ from $\frac{3}{4x}$.

The L.C.M. of the denominators is $20x$.

Expressing each fraction with denominator $20x$, we have

$$\begin{aligned} \text{(i)} \quad \frac{3}{4x} + \frac{1}{10x} &= \frac{15}{20x} + \frac{2}{20x} \\ &= \frac{15+2}{20x} = \frac{17}{20x}. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \frac{3}{4x} - \frac{1}{10x} &= \frac{15}{20x} - \frac{2}{20x} \\ &= \frac{15-2}{20x} = \frac{13}{20x}. \end{aligned}$$

4. Simplify :

$$(i) \frac{6ab^3c^4}{10a^2b^2c^3}$$

$$(ii) \frac{15a^2x^2y}{5axy^2}$$

$$(iii) \frac{48a^2x^2y^5}{12a^2x^3y^4}$$

$$(iv) \frac{13a^2b^2c^4}{39a^3b^3c^5}$$

$$(v) \frac{51x^5y^3z^2}{17x^4y^2z}$$

$$(vi) \frac{21a^7x^7y^3}{7a^8x^6y^5}$$

5. Fill in the gaps in the following :—

$$(i) \frac{5x}{2y} = \frac{\quad}{14y^2}$$

$$(ii) \frac{4a^2b}{6ab^2} = \frac{\quad}{18a^2b^2} = \frac{\quad}{3b}$$

$$(iii) a^3 = \frac{\quad}{b^2}$$

$$(iv) \frac{1}{a} = \frac{b^2}{\quad}$$

$$(v) \frac{m^2}{a} = \frac{m^3n^3}{\quad}$$

6. Reduce the following to fractions having the least common denominator :

$$(i) \frac{5}{6a}, \frac{4}{3x}, \frac{3c}{ax}$$

$$(ii) \frac{3}{4x}, \frac{x}{6y^2}, \frac{y}{3x^2}$$

$$(iii) \frac{x^2}{2ab}, \frac{y^2}{3ac}, \frac{z^2}{4bc}$$

$$(iv) \frac{x^2y}{a^3}, \frac{2x^3}{3a^2b}, \frac{3y^3}{4ab^2}, \frac{4xy^2}{5b^3}$$

7. Simplify the following fractions :

$$(i) \frac{2x}{3} + \frac{3x}{4}$$

$$(ii) \frac{3a}{2x} + \frac{b}{x}$$

$$(iii) \frac{1}{2x} + \frac{1}{3x}$$

$$(iv) \frac{x}{y} + \frac{y}{x}$$

$$(v) 1 + \frac{x}{y}$$

$$(vi) \frac{1}{x} + 3$$

$$(vii) 1 - \frac{1}{3x}$$

$$(viii) \frac{a}{b} - \frac{2a}{3b}$$

$$(ix) 1 - \frac{a}{b}$$

$$(x) 1 - \frac{1}{xy}$$

$$(xi) \frac{2}{xy} - \frac{xy}{z}$$

$$(xii) \frac{x}{2} + \frac{x}{3} + \frac{x}{4}$$

$$(xiii) \frac{x}{a} + \frac{y}{b} + \frac{z}{c}$$

$$(xiv) \frac{2x}{3} - \frac{x}{4} + \frac{3x}{5}$$

$$(xv) \frac{a}{2x} + \frac{a}{3x} + \frac{a}{4x}$$

$$(xvi) \frac{2a}{3x} + \frac{5b}{12y} + \frac{3c}{4z}$$

$$(xvii) \frac{x^2}{12} - \frac{3xy}{15} + \frac{2y^2}{18}$$

$$(xviii) \frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}$$

$$(xix) \frac{b-c}{bc} + \frac{c-a}{ca} + \frac{a-b}{ab}$$

8. Simplify the following:

$$(i) \frac{a^2}{bc} \times \frac{b^2}{ac}$$

$$(ii) xz \div \frac{x}{z}$$

$$(iii) \frac{a}{b} \times \frac{b}{c} \times \frac{c}{a}$$

$$(iv) \frac{a^2b^2}{c} \times \frac{c^2}{ab}$$

$$(v) \frac{12bc}{35a^2} \times \frac{21ab^2}{30b^3}$$

$$(vi) \frac{36x^2y^2}{15z^3} \div \frac{27xy^2}{10z^2}$$

9. What fraction is $4m$ inches of m feet?

10. If in the fraction $\frac{4x}{5x}$ different values are given to x , is its value changed?

11. ABC is a straight line, its part AB measures x units and BC measures y units. What fraction is (i) AB of AC , (ii) AC of BC ?

5. Generalisation, Substitution, Problems.

It is important to remember that the *letters* stand for *numbers* and not for the *number of things*. Thus it is wrong to say 'Let l stand for the length of a rectangle'; l can stand for the number of linear units in the length and not for the length itself.

The term *quantity* means *number of things*. Thus x *rupees* is a quantity, but x is a number. When a quantity is to be indicated, the unit must be stated.

EXERCISE 5.

Criticise the following statements:

1. If one chair costs x , then four chairs cost $4x$.
2. If the length of a rectangle is l ft. and breadth b ft., its area is lb .
3. The price of silk per yd. is x .
4. The temperature of a place was x and then it rose 5 degrees; thus it was then $x + 5$.
5. Let a chair cost x and a table y , then since the price of the table is Rs. 5 more than that of the chair, $\therefore y - x = \text{Rs. } 5$.

Example 1. Suppose we know that

$$\text{the sum of the first 2 natural numbers} = \frac{2(1+2)}{2};$$

$$\text{,, ,, 3 ,,} = \frac{3(1+3)}{2};$$

$$\text{,, ,, 4 ,,} = \frac{4(1+4)}{2};$$

and so on.

From the similarity of the form of these expressions, we can deduce a general form:

$$\text{the sum of the first } n \text{ natural numbers} = \frac{n(1+n)}{2},$$

which includes all the above three statements.

The process by which we arrive at the general form is called **generalisation**.

Generalise:

6. 5 exceeds 3 by $5-3$
 7 ,, 4 by $7-4$
 9 ,, 5 by $9-5$.

7. $4 + 5 = 5 + 4$
 $7 + 3 = 3 + 7$
 $9 + 4 = 4 + 9$.

8. $5 \times 4 = 4 \times 5$
 $7 \times 9 = 9 \times 7$
 $12 \times 8 = 8 \times 12$.

9. $7^2 \times 7 = 7^3$
 $9^2 \times 9 = 9^3$
 $11^2 \times 11 = 11^3$.

10. $9 \times \frac{1}{9} = 1$
 $7 \times \frac{1}{7} = 1$
 $12 \times \frac{1}{12} = 1$.

11. $0 \times 3 = 0$
 $0 \times 4 = 0$
 $0 \times 5 = 0$.

12. $3 \times 0 = 0$
 $4 \times 0 = 0$
 $5 \times 0 = 0$

13. 2×1 is an even number
 2×2 ,, ,,
 2×3 ,, ,,

14. $2 \times 1+1$ is an odd number
 $2 \times 2+1$,, ,,
 $2 \times 3+1$,, ,,

15. The sum of the first 2 odd numbers is 2^2

„ „ 3 „ „ 3^2

„ „ 4 „ „ 4^2

16. 4 times $6 + 6 = 5$ times 6

4 times $7 + 7 = 5$ times 7

4 times $9 + 9 = 5$ times 9.

17. 5% of a number is $\frac{5}{100}$ of that number

7% „ „ $\frac{7}{100}$ „ „

9% „ „ $\frac{9}{100}$ „ „

18. In a rectangle, if

length = 16 ft., breadth = 12 ft., area = 16×12 sq. ft.

„ = 24 ft., „ = 18 ft., „ = 24×18 „ „

„ = 15 ft., „ = 14 ft., „ = 15×14 „ „

19. $9 - 1$, 9, $9 + 1$ are three consecutive numbers

$7 - 1$, 7, $7 + 1$ „ „ „ „

$28 - 1$, 28, $28 + 1$ „ „ „ „

20. One number is ϕ . (a) Give the next higher consecutive number. (b) Give the next lower consecutive number.

21. Out of three consecutive numbers the middle one is m ; give the other numbers.

Substitution. It is extremely useful to derive a few particular cases from a general statement by substituting the given numerical values for the letters, as illustrated in the next example.

Example 2. If a rectangular room is l ft. long, b ft. broad and h ft. high, the area of its four walls is $2(l + b) \times h$ sq. ft. Derive from this statement the area of the four walls of a room, 16 ft. long, 14 ft. wide and 12 ft. high.

Since in this case $l = 16$, $b = 14$ and $h = 12$,

∴ the area of the four walls

$$= 2 (16 + 14) \times 12 \text{ sq. ft.}$$

$$= 2 \times 30 \times 12 \text{ sq. ft.}$$

$$= 720 \text{ sq. ft.}$$

22. If the parallel sides of a trapezium are a and b and its altitude is h , the area $= \frac{1}{2} (a + b) \times h$. Derive from this the area of another trapezium whose parallel sides are 18 ft. and 12 ft. and the altitude is 8 ft.

23. From the general statement "area of a circle $= \pi r^2$ ", where r stands for the radius of the circle, derive the area of a circle whose radius is 15 ft.

24. If the interest on a certain sum p @ the rate $r\%$ per annum for t years is $\frac{p \times r \times t}{100}$, find the interest on Rs. 450 @ 6% per annum for 3 years.

25. If $a + 1$ is the square root of $a^2 + 2a + 1$, show that it is true when $a = 5, 7, 9, 11$.

26. The temperature of a place is x degrees Centigrade or $\frac{9x}{5} + 32$ degrees Fahrenheit. What will be its temperature in Fahrenheit, when it is 40 degrees Centigrade?

27. If a figure has n sides, the sum of its interior angles is $(2n - 4)$ right angles. What is the sum of the angles of (i) a figure of 5 sides, (ii) a figure of 8 sides, (iii) a figure of 10 sides and (iv) a figure of 12 sides?

28. If a figure is regular and has n sides, each of its interior angles is $\frac{2n - 4}{n}$ right angles. What is the magnitude of an interior angle of (i) a six-sided regular figure, (ii) an eight sided regular figure?

Hint. If you fail to solve a literal problem, frame for yourself a similar problem using numbers for letters and solve it; then do the similar process for the given problem, as illustrated further in examples 3 and 4.

Example 3. The telegraph posts are fixed d ft. apart along a railway line and a train passes n poles per minute. Find the speed of the train per hour.

Let us frame a similar problem, putting numbers instead of the letters d and n .

The telegraph posts are fixed 220 ft. apart along a railway line and a train passes 9 poles per minute. Find the speed of the train per hour.

In one minute the train passes 9 poles or covers $(9 - 1)$ or 8 distances between two consecutive poles.

$$\begin{aligned} \therefore \text{ in 1 minute it goes } & 220 \times 8 \text{ ft.} \\ \therefore \text{ „ 60 minutes } & \text{ „ } 220 \times 8 \times 60 \text{ ft.} \\ \therefore \text{ „ 1 hour } & \text{ „ } \frac{220 \times 8 \times 60}{3 \times 1760} \text{ miles.} \end{aligned}$$

Now we can work out the original problem:

In one minute the train passes n poles or covers $(n - 1)$ distances between two consecutive poles.

$$\begin{aligned} \therefore \text{ in 1 minute it goes } & d \times (n - 1) \text{ ft.} \\ \therefore \text{ „ 60 minutes } & \text{ „ } d \times (n - 1) \times 60 \text{ ft.} \\ \therefore \text{ „ 1 hour } & \text{ „ } \frac{d \times (n - 1) \times 60}{3 \times 1760} \text{ miles} \\ \therefore \text{ „ 1 „ } & \text{ „ } \frac{d(n - 1)}{88} \text{ miles.} \end{aligned}$$

Example 4. A cow is bought for x rupees and sold at a profit for y rupees; find the profit per cent.

Let us take a similar problem in Arithmetic. A cow is bought for Rs. 57 and sold for Rs. 72; find the profit per cent.

$$\begin{aligned} \text{The profit is} & \text{Rs. } (72 - 57), \\ \text{The cost price is} & \text{Rs. } 57, \\ \therefore \text{ the profit is} & \left[\frac{(72 - 57)}{57} \times 100 \right] \% \end{aligned}$$

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Now in the original problem,

the profit is Rs. $(y - x)$,

the cost price is Rs x ,

\therefore the profit is $\left[\frac{(y - x)}{x} \times 100 \right] \%$

i.e. $\frac{100(y - x)}{x} \%$

29. Reduce Rs. x , y annas to annas.
30. How many hours does it take to walk :
(i) x miles at the rate of 5 miles an hour ?
(ii) x miles at the rate of y miles an hour ?
31. In t hours a man walks m miles :
(i) At what rate does he walk in miles per hour ?
(ii) How many yards does he walk in 15 minutes ?
32. A train runs with the speed of m miles per hour ; what is its speed in yards per minute ?
33. A clock loses s seconds in 24 hours ; in how many days will it be x minutes too slow ?
34. If 1s. = 10 as., how many shillings can we have for Rs. 25 ? for Rs. x ?
35. If three-fourths of the distance between Delhi and Simla is x miles, what is the distance between the two towns ?
36. If a yds. of silk cost Rs. 15 ; how much of it can be had for Rs. x ?
37. For x miles the railway fare is Rs. y ; how much will a journey of
(i) $8x$ miles, (ii) m miles,
cost at the same rate ?
38. If x men can do a piece of work in 14 days, how many days will $6x$ men take to do it ? How long will y men take to do it ?

39. A garrison has provisions for x men for y days. If m more men come in, how long will these last ?
40. If 15 srs. of sugar cost Rs. 4 as. 10, how much of it can be bought for Rs. x as. y ?
41. A merchant bought sugar worth Rs. m and sold it at a profit for Rs. n . Find his profit per rupee.
42. A motor car runs l miles in a hours on the first day, m miles in b hours on the second day and n miles in c hours on the third day. Find its average speed per hour.
43. What is the result of increasing x by 16 per cent. ?
44. How much per cent. is x of y ?
45. Express as a percentage (i) $\frac{1}{a}$, (ii) $\frac{a}{b}$.
46. Find the simple interest on Rs. 300 at $r\%$ per annum for 5 years.
47. Find the simple interest on Rs. p at $r\%$ per annum for 5 years.
48. Find the simple interest on Rs. p at $r\%$ per annum for t years.
49. If $x\%$ of a property is spent every year, how long will the whole property last ?
50. If x seeds are sown and 84% of them take root, how many of them are wasted ?
51. Sugar is bought at a as. per seer and sold at a profit of $12\frac{1}{2}\%$. Find the selling price of 6 seers.
52. The price of cycles is reduced by 5%. If the old price of a cycle were r rupees, what is its new price ?
53. A bookseller allows $x\%$ commission on the published price of his books sold for cash. How many annas in a rupee is this ?

54. The price of wheat has fallen $p\%$ in two years. It is now selling @ 32 seers per rupee. What was its rate two years ago?
55. A rectangular path is l yds. long and 5 ft. wide; find its area in square feet.
56. Find the cost in rupees of a carpet for a room x ft. long and y ft. wide @ 4 as. per sq. ft.
57. The area of a wall is A sq. ft. and its height is 8 ft.; find its length.
58. A piece of cloth is x yds. long and y'' wide. How many pieces $10'' \times 8''$ can be cut out of it?
59. What is the volume of a brick a'' long, b'' wide and c'' thick? What is the area of its surface?
60. A reservoir is l ft. long and b ft. wide. If its capacity is x cu. ft., find its depth.
61. The cross section of a cylinder is x sq. inches and its length is 50". Find its volume.
62. Find the number of bricks $9'' \times 4\frac{1}{2}''$ required for a rectangular floor x ft. long and y ft. wide.
63. Find the number of bricks $9'' \times 4\frac{1}{2}'' \times 3''$ required for building a rectangular platform a ft. long, b ft. wide and 30" high.
64. Find the number of bricks $9'' \times 4\frac{1}{2}'' \times 3''$ required to build a wall h ft. high, w'' thick round a rectangular garden l ft. long and b ft. wide.
65. The thickness of a sheet of paper is $\frac{1}{k}$ in. How many sheets will be required to form a pile of paper h in. high?
66. Find the length of a wire whose cross-section is $\frac{1}{t}$ sq. in. and whose volume is 1 cu. ft.

CHAPTER II

DIRECTED NUMBERS, FUNDAMENTAL LAWS, FIRST FOUR RULES

1. Illustrations.

(i) If a tradesman invests in business a capital of Rs. 100, and from his first transaction he gains Rs. 5, and from his second transaction he loses Rs. 5, his capital or financial position remains unchanged.

Here the *loss* of Rs. 5 *destroys* the *gain* of Rs. 5 ; thus the loss of Rs. 5 and the gain of Rs. 5 are *opposite in character* but *equal in magnitude*.

(ii) If a man ascends 10 yds. and then descends 10 yds., his ultimate position remains unchanged.

Here the *descent* of 10 yds. *destroys* the *ascent* of 10 yds. ; thus the ascent of 10 yds. and the descent of 10 yds. are *opposite in character* but *equal in magnitude*.

(iii) If a man walks 3 miles east and then 3 miles west, his ultimate position remains unchanged.

Here the effect of *going 3 miles east* is *destroyed* by the effect of *going 3 miles west* ; thus going 3 miles east and going 3 miles west are *opposite in character* but *equal in magnitude*.

(iv) $7 + 3 - 3 = 7$.

Here the effect of *adding 3* is *destroyed* by the effect of *subtracting 3*, and thus the subtraction of 3 and the addition of 3 are *opposite operations* with *equal magnitudes*.

These four examples are of *equal* and *opposite quantities* or *movements*.

It is evident that the gain of Rs. 8 does not *completely destroy* the loss of Rs. 12, but the effect of one on the

capital is opposite to the effect of the other ; hence *gain* and *loss* are two *opposite quantities*.

Similarly, credit and debit, income and expenditure, ascent and descent, rise and fall, the distance to the right-hand side and the distance to the left-hand side, the distance to the east and the distance to the west, the distance to the north and the distance to the south, the degrees north of the equator and the degrees south of the equator, the degrees above the freezing-point and the degrees below the freezing-point, the arrival of men and the departure of men, the time before a particular event and the time after that event, the operation of addition and the operation of subtraction, are pairs of opposite quantities or movements.

In Arithmetic the operations of addition and subtraction which are opposite to each other, are denoted by the signs $+$ and $-$ respectively ; similarly, it would be *convenient* to denote any pair of opposite quantities or movements by the same signs, $+$ and $-$.

If one quantity or movement be represented by the sign $+$, the opposite one would be represented by the sign $-$.

EXERCISE 6. (Oral)

1. Name three pairs of equal and opposite quantities.
2. Name two pairs of equal and opposite movements.
3. If a loss of Rs. 5 be denoted by $+$ Rs. 5, what would be the notation for a gain of Rs. 12 ?
4. If 30 miles towards east be represented by -30 miles, what would be the notation for 20 miles towards west ?
5. If 60 miles towards north be denoted by $+$ 60 miles, what would be the notation for 15 miles towards south ?
6. If 23° north of the equator be denoted by $+23^{\circ}$, what would be the notation for 17° south of the equator ?

7. If 500 yrs, before the birth of Christ be denoted by $+500$ yrs., what would be the meaning of -500 yrs.?
8. If the arrival of 5 men be denoted by $+5$ men, what would be the meaning of -5 men?
9. If $-Rs. 50$ mean a gain of Rs. 50, what would be the meaning of $+Rs. 60$?
10. If $+Rs. 12$ denote Rs. 12 received, what would $-Rs. 12$ denote?
11. If 4000 ft. height above the sea-level be represented by $+4000$ ft., what would be the meaning of -2000 ft.?
12. What must be taken as negative, if each of the following is taken as positive :
 - (i) Number of feet to the right-hand side.
 - (ii) Number of miles to the east.
 - (iii) Number of rupees lost.
 - (iv) Degrees of falling temperature.
 - (v) Degrees north of the equator.
 - (vi) Days to come.

2. **Convention.** We have seen that if *either* of the two opposite quantities is denoted by the sign $+$, the other is denoted by the sign $-$; but, in general, it is *usual* to mark such quantities *positive* as *raise* or *strengthen* our position, and to mark such quantities *negative* as *lower* or *weaken* our position.

Thus, *gain, income, credit, ascent, rise*, etc., are taken as positive, and *loss, expenditure, debit, descent, fall*, etc., are taken as negative.

Similarly, motion and distance to the right-hand, motion and distance to the east, motion and distance to the north, degrees north of the equator, degrees above the freezing-point, the coming of men, etc., are taken as positive; whereas motion and distance to the left-hand, motion and distance

to the west, motion and distance to the south, degrees south of the equator, degrees below the freezing point, the departure of men, etc., are taken as negative.

NOTE. The + sign is generally omitted before a positive quantity; hence when no sign is prefixed to a quantity, + sign is understood. Thus + Rs. 5 and Rs. 5, + 8 and 8 are positive.

EXERCISE 7. (Oral)

Write the following symbolically prefixing the *usual* signs.

1. An income of Rs. 50.
2. A loss of Rs. 20.
3. A debt of Rs. 100.
4. An expenditure of Rs. 40.
5. A saving of Rs. 15.
6. An ascent of 25 yds.
7. A fall of 125 yds.
8. A promotion of Rs. 5.
9. 200 ft. below the sea level.
10. 500 yds. above the sea level.
11. 13 miles due east.
12. 15 miles due north.
13. 17 miles due south.
14. 20 miles due west.
15. 20° north of the equator.
16. 15° south of the equator.
17. 8 men arrived.
18. 10 men gone away.
19. 13 yds. to the left-hand.
20. 19 yds. to the right-hand.
21. 35° above the freezing-point.
22. 23° below the freezing-point.
23. Rs. 40 received.
24. Rs. 40 given away.
25. Express with proper signs all the following quantities as gains:
 - (i) gain of Rs. 12;
 - (ii) loss of Rs. 9;
 - (iii) gain of Rs. 25;
 - (iv) loss of Rs. 15.
26. Express all the following, first as northerly movements and then as southerly movements:
 - (i) $5^{\circ}N.$;
 - (ii) $7^{\circ}S.$;
 - (iii) $3^{\circ}N.$;
 - (iv) $8^{\circ}S.$

27. If the dates are reckoned + from 0 A. D. onwards, express the following with proper signs:

(i) 420 A.D.;

(ii) 75 B.C.;

(iii) 1625 A.D.;

(iv) 1200 B.C.

3. Definitions. (i) Gain and loss, credit and debit, etc., are reckoned from the original financial position of a man.

(ii) Rise and fall, ascent and the descent, etc., are measured from the level of the ground.

(iii) The distance to the east and the distance to the west, the distance to the right-hand side and the distance to the left-hand side, etc., are measured from a particular position.

Similarly, in other cases as well, we always measure the magnitudes of opposite quantities from a particular position.

The position from which we measure the magnitude of opposite quantities is called the **starting-point** and numbers indicating their measures and direction are called **directed numbers**.

Example 1. A barrel floating on the surface of a sea is carried by successive tides as follows :

3 miles east, 4 miles west, 2 miles east, 5 miles west, 6 miles east. Represent these movements by means of the usual signs.

Since the distance to the east is represented by the positive sign and the distance to the west is represented by the negative sign, therefore we have :

$$(+3 \text{ miles}) + (-4 \text{ miles}) + (+2 \text{ miles}) + (-5 \text{ miles}) \\ + (+6 \text{ miles}),$$

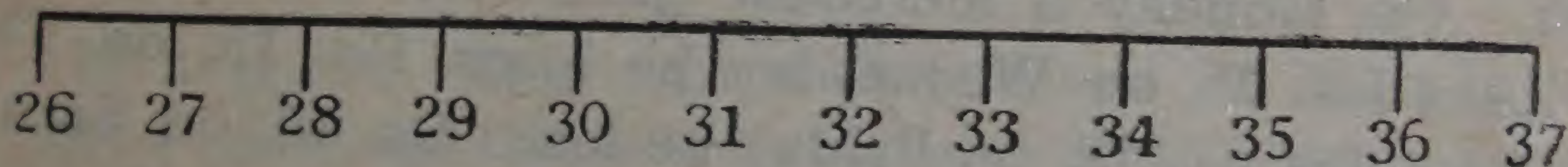
as the representation of the changes in the position of the barrel.

Example 2. A person gains Rs. 30, loses Rs. 25, gains Rs. 15, and then loses Rs. 10. Represent these changes symbolically, using a for Rs. 5.

Since a is to stand for Rs. 5, therefore a gain of Rs. 30 would be represented by $+6a$, loss of Rs. 25 would be represented by $-5a$, a gain of Rs. 15 would be represented by $+3a$, and a loss of Rs. 10 would be represented by $-2a$. Thus these changes, when represented symbolically, stand as follows :

$$(+6a) + (-5a) + (+3a) + (-2a).$$

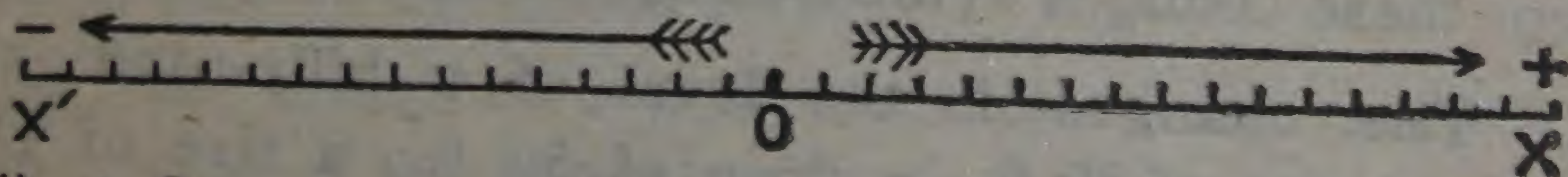
Relative Position. Suppose all measurements are calculated on the following scale from the 32 mark.



Since 35 is 3 steps forward from 32, it is said to be $(+3)$ *relative* to 32, and as 30 is 2 steps backward from 32, it is said to be (-2) *relative* to 32.

EXERCISE 8.

1. In the graduated scale XX' , take O as the starting-point, and *mark* on it the positions of the following points :



- | | | |
|--------------|----------------|---------------|
| (i) $+3$. | (ii) -3 . | (iii) $+5$. |
| (iv) -8 . | (v) -10 . | (vi) $+12$. |
| (vii) -9 . | (viii) $+15$. | (ix) $+7$. |
| (x) 0 . | (xi) $+14$. | (xii) -15 . |

2. Tell from the scale the number of divisions between

- | | |
|-------------------------|-----------------------|
| (i) $+3$ and -3 . | (ii) $+5$ and -8 . |
| (iii) -10 and $+10$. | (iv) -5 and -8 . |
| (v) -6 and $+4$. | (vi) 0 and -7 . |
| (vii) -9 and 0 . | (viii) $+8$ and 0 . |
| (ix) 0 and $+5$. | |

How many divisions do I move, and in what direction

3. If I move from $+3$ to -3 ?
4. If I move from $+5$ to $+8$?
5. If I move from -8 to -5 ?
6. If I move from -5 to $+8$?
7. If I move from $+9$ to $+4$?
8. If I move from $+11$ to -3 ?
9. If I move from -10 to $+2$?
10. If I move from -4 to -10 ?
11. If I move from $+3$ to -9 ?

12. On Monday a merchant gains Rs. 60, on Tuesday he loses Rs. 25, on Wednesday he loses Rs. 15, on Thursday he gains Rs. 12, on Friday he loses Rs. 16, on Saturday he gains Rs. 24, and on Sunday he gains Rs. 40. Represent these changes by means of the usual signs.

13. The daily changes in the height of a barometer are: Monday a fall of $\cdot 6$ in., Tuesday a rise of $\cdot 8$ in., Wednesday a fall of $\cdot 4$ in., Thursday a rise of $\cdot 2$ in., Friday a rise of $\cdot 3$ in., Saturday a fall of $\cdot 2$ in., and Sunday a fall of $\cdot 5$ in. Represent these changes symbolically, using x for $\cdot 1$ in.

14. The changes in the level of a hilly-road are as follows: a rise of 80 ft., a drop of 50 ft., a rise of 110 ft., a drop of 60 ft., a rise of 130 ft., a drop of 70 ft., a rise of 150 ft. Represent the level symbolically, using x for 10 ft.

15. P, Q, R, S, T are five stations on a railway. From P to Q the level falls 50 ft., from Q to R it rises 70 ft., from R to S it falls 40 ft., and from S to T it rises 80 ft. Represent the whole level symbolically, using a for 5 ft.

16. An aeroplane rises to a height of 850 ft., then descends 350 ft., it again rises 260 ft., and then descends 145 ft. Express its ascent and descent symbolically.

17. The average temperature of a day was 80° . At certain hours of the day the temperature was 75° , 78° , 84° , 88° . State each of these, relative to the average temperature.

18. Reckoning times from 10 A.M., express each of the following times on a certain day by the directed numbers :

(i) 4 P.M., (ii) noon, (iii) 7-30 A.M., (iv) 3-20 P.M.

19. The freezing-point of a Fahrenheit thermometer is 32° . Express the following temperatures relative to the freezing-point :

(i) 78° , (ii) 24° , (iii) 32° , (iv) 0° .

20. If a goods train runs with a speed of 20 miles an hour and a passenger train with a speed of 35 miles an hour in the same direction, express :

(i) the speed of the passenger train relative to that of the goods train ;

(ii) the speed of the goods train relative to that of the passenger train ;

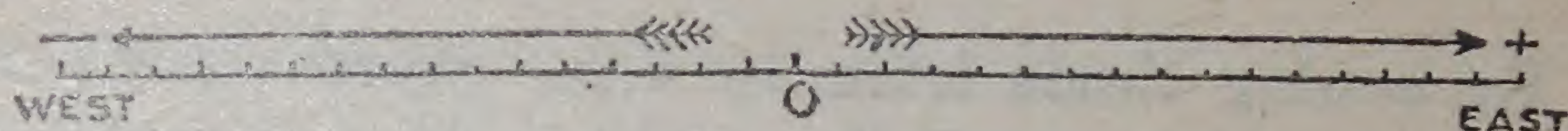
(iii) the speed of a pole fixed along the railway line relative to that of the goods train :

(iv) the speed of a pole fixed along the railway line relative to that of the passenger train.

ADDITION

1. **Graphic Illustrations.** The following graduated scale stands for a straight road from east to west or west to east ; each of its divisions stands for *one mile*. *O* is the starting-point. A man is supposed to be travelling along this road.

Mark the successive positions of the man and express in each of the following cases the whole movement by means of the *usual signs* :



(i) When he goes 8 miles to the east, then 5 miles to the east, and is thus 13 miles to the east.

Symbolic representation : $(+8 \text{ miles}) + (+5 \text{ miles}) = +13 \text{ miles}$.

(ii) When he goes 4 miles to the east, 3 miles to the east, and then 5 miles in the same direction, and is thus 12 miles to the east.

Symbolic representation : $(+4 \text{ miles}) + (+3 \text{ miles})$
 $+ (+5 \text{ miles}) = +12 \text{ miles}$.

(iii) When he goes 8 miles to the west, then 5 miles to the west, and is thus 13 miles to the west.

Symbolic representation : $(-8 \text{ miles}) + (-5 \text{ miles})$
 $= -13 \text{ miles}$.

(iv) When he goes 4 miles to the west, 3 miles to the west, and then 5 miles in the same direction, and is thus 12 miles to the west.

Symbolic representation : $(-4 \text{ miles}) + (-3 \text{ miles})$
 $+ (-5 \text{ miles}) = -12 \text{ miles}$.

Money Illustrations. Let a stand for *one rupee*.

Arithmetical examples	Algebraic representation
(i) Rs. 4 gain + Rs. 3 gain = Rs. 7 gain.	$(+4a) + (+3a) = +7a$.
(ii) Rs. 4 gain + Rs. 3 gain + Rs. 5 gain = Rs. 12 gain.	$(+4a) + (+3a) + (+5a)$ $= +12a$.
(iii) Rs. 4 loss + Rs. 3 loss = Rs. 7 loss.	$(-4a) + (-3a) = -7a$.
(iv) Rs. 4 loss + Rs. 3 loss + Rs. 5 loss = Rs. 12 loss.	$(-4a) + (-3a) + (-5a)$ $= -12a$.

From these and similar examples it follows :

Rule 1. The sum of a number of like terms, all of the same sign, is a single like term of the same sign, and its co-efficient is the sum of the numerical values of the several co-efficients.

Example 1. Find the sum of $+2ab$, $+7ab$, $+8ab$.

Here all the terms are alike and have the same sign; hence the literal part of the sum is ab , the co-efficient is 17, i.e. $(2+7+8)$, and the sign to be prefixed is $+$.

$$\therefore \text{sum} = +17ab.$$

Example 2. Find the sum of $-3ab^2$, $-5ab^2$, $-7ab^2$.

Here all the terms are alike and have the same sign; hence the literal part of the sum is ab^2 , the co-efficient is 15, i.e. $(3+5+7)$, and the sign to be prefixed is $-$.

$$\therefore \text{sum} = -15ab^2,$$

In the above scale, mark the successive positions of the man and express in each of the following cases the whole movement by means of the *usual signs*:

(v) When he goes 8 miles to the east, then 5 miles to the west, and is thus 3 miles to the east.

$$\begin{aligned} \text{Symbolic representation : } (+8 \text{ miles}) + (-5 \text{ miles}) \\ = +3 \text{ miles.} \end{aligned}$$

(vi) When he goes 8 miles to the west, then 5 miles to the east, and is thus 3 miles to the west.

$$\begin{aligned} \text{Symbolic representation : } (-8 \text{ miles}) + (+5 \text{ miles}) \\ = -3 \text{ miles.} \end{aligned}$$

(vii) When he goes 8 miles to the east, 5 miles to the west, then 7 miles to the west, and is thus 4 miles to the west.

$$\begin{aligned} \text{Symbolic representation : } (+8 \text{ miles}) + (-5 \text{ miles}) \\ + (-7 \text{ miles}) = -4 \text{ miles.} \end{aligned}$$

(viii) When he goes 8 miles to the west, 5 miles to the east, then 7 miles to the east, and is thus 4 miles to the east.

$$\begin{aligned} \text{Symbolic representation : } (-8 \text{ miles}) + (+5 \text{ miles}) \\ + (+7 \text{ miles}) = +4 \text{ miles.} \end{aligned}$$

Money Illustrations. Let a stand for *one* rupee.

Arithmetical examples	Algebraic representation
(i) Rs. 8 gain + Rs. 3 loss = Rs. 5 gain.	$(+8a) + (-3a) = +5a.$
(ii) Rs. 8 loss + Rs. 3 gain = Rs. 5 loss.	$(-8a) + (+3a) = -5a.$
(iii) Rs. 8 gain + Rs. 3 loss + Rs. 4 gain = Rs. 9 gain.	$(+8a) + (-3a) + (+4a)$ $= +9a.$
(iv) Rs. 8 loss + Rs. 3 gain + Rs. 4 loss = Rs. 9 loss.	$(-8a) + (+3a) + (-4a)$ $= -9a.$

From these and similar examples it follows:

Rule 2. The sum of a number of like terms, not all of the same sign, is a single term. To obtain its co-efficient add together the co-efficients of the positive terms and the co-efficients of the negative terms, take the difference of these two results, and prefix the sign of the greater.

Example 3. Find the sum of

$$+6ab^2c - 4ab^2c + 8ab^2c - 12ab^2c.$$

Here all the terms are alike, but have different signs; hence the letters in the sum are ab^2c , the co-efficient is 2, i.e.

$(6+8-4+12)$, and the sign to be prefixed is —.

$$\text{sum} = -2ab^2c.$$

EXERCISE 9.

1. Prove graphically that:

- | | |
|----------------------------|----------------------------|
| (i) $(+7) + (+12) = +19,$ | (ii) $(-7) + (-12) = -19,$ |
| (iii) $(+7) + (-12) = -5,$ | (iv) $(-7) + (+12) = +5,$ |
| (v) $+5 - 3 = +2,$ | (vi) $-5 + 3 = -2,$ |
| (vii) $+8 - 8 = 0,$ | (viii) $-12 + 12 = 0,$ |
| (ix) $+5 - 7 + 6 = +4,$ | (x) $-5 + 7 - 6 = -4.$ |

Add (orally) :

$$\begin{array}{r} 2. \quad -7x \\ +4x \\ \hline \end{array}$$

$$\begin{array}{r} 3. \quad 5ax \\ -7ax \\ \hline \end{array}$$

$$\begin{array}{r} 4. \quad +19axy \\ -9axy \\ \hline \end{array}$$

$$\begin{array}{r} 5. \quad +8ab \\ -7ab \\ +5ab \\ \hline \end{array}$$

$$\begin{array}{r} 6. \quad -4xyz \\ +7xyz \\ -13xyz \\ \hline \end{array}$$

$$\begin{array}{r} 7. \quad +9x^2y \\ -5x^2y \\ +12x^2y \\ \hline \end{array}$$

$$\begin{array}{r} 8. \quad -5y^2z \\ -8y^2z \\ 9y^2z \\ \hline \end{array}$$

$$\begin{array}{r} 9. \quad -2m^2n \\ +17m^2n \\ -5m^2n \\ \hline \end{array}$$

$$\begin{array}{r} 10. \quad +8p^2q^2 \\ -12p^2q^2 \\ +4p^2q^2 \\ \hline \end{array}$$

Simplify :

$$11. \quad 4ab + 3ab - 2ab - 7ab.$$

$$12. \quad 13xyz - 4xyz + 5xyz.$$

$$13. \quad -5abx^2 + 8abx^2 - 9abx^2 + 4abx^2 - 2abx^2.$$

$$14. \quad -7pq^2r - 12pq^2r + 7pq^2r - 3pq^2r + 5pq^2r - 8pq^2r.$$

Simplify by collecting like terms :

$$15. \quad a - 2b + 3c - 4a + 6b - 5c + 2a.$$

$$16. \quad x - 4y + 5z - 3y + 4z - 5x + 2y - 10z + 5y + x.$$

$$17. \quad 3p^2 - 3pq + 4q^2 - p^2 + 7pq - 6p^2 - 2pq + 5q^2.$$

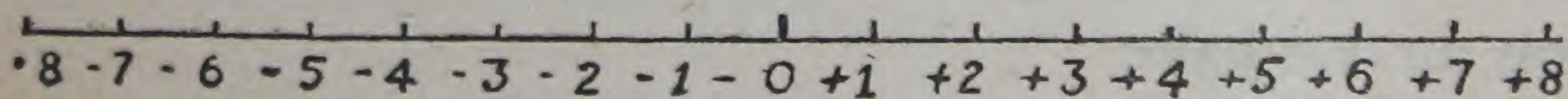
$$18. \quad 5ab - 4bc + ca - 2ab + 6ca + 7bc - 3ca + bc.$$

$$19. \quad 6x^2yz + xy^2z - 4x^2yz - 5xyz^2 + 3xy^2z + 7xyz^2.$$

$$20. \quad 4ax^2y - 5a^2xy + 8ax^2y - 7axy^2 + 2a^2xy - 4axy^2.$$

* *Def.* In a scale of directed numbers, the number midway between any two of the numbers is called their **Arithmetic mean**.

Thus in this scale



- (i) the Arithmetic mean of +1 and +7 is +4,
- (ii) the Arithmetic mean of +2 and -4 is -1,
- (iii) the Arithmetic mean of -2 and -6 is -4.

But (i) half the sum of $+1$ and $+7$ is $+4$,

(ii) half the sum of $+2$ and -4 is -1 ,

(iii) half the sum of -2 and -6 is -4 .

therefore, the Arithmetic mean of two directed numbers may be obtained by dividing their algebraic sum by 2.

Similarly, the Arithmetic mean or the average of, say 8 directed numbers = $\frac{\text{their algebraic sum}}{8}$.

* 21. Find the Arithmetic mean of the following directed numbers :

(i) $+16$ and $+24$,

(ii) -8 and $+14$,

(iii) -8 and -20 ,

(iv) $+8$ and -22 ,

(v) -7 , $+4$, -5 , $+6$,

(vi) -9 , $+5$, $+3$, -6 , $+7$.

* 22. Draw a straight line XOX' on squared paper and represent on it 5 points whose heights are $+8$, $+4$, -2 , $+3$, -3 . Find the average height of these points; draw a straight line LM at the average height above XOX' . Express the height of these points relative to the straight line LM .

* 23. Represent on squared paper 9 points whose heights are $+10$, -8 , $+2$, -7 , $+3$, $+5$, -4 , $+6$, $+11$, with reference to a st. line AB .

Draw another st. line parallel to AB and at the average height of these points. Express the height of each of these points relative to this st. line.

Example 4. Add together $5a^3 - 4a^2b + 6ab^2 - 2b^3$, $a^3 - 2ab^2 - 3a^2b$, and $b^3 - 3a^3 + ab^2$.

Here, by arranging the like terms in columns and adding them up, we have

$$\begin{array}{r}
 5a^3 - 4a^2b + 6ab^2 - 2b^3 \\
 a^3 - 3a^2b - 2ab^2 \\
 -3a^3 \qquad \qquad + ab^2 + b^3 \\
 \hline
 3a^3 - 7a^2b + 5ab^2 - b^3
 \end{array}$$

Add together :

24. $x^2 + 3xy, -2xy + y^2$. 25. $x^2 + xy, -xy + y^2$.
 26. $a^2 + ab, -ab - b^2$. 27. $p^2 - 3pq + q^2, -5p^2 + 6pq + 2q^2$.
 28. $a^2 + a + 1, 3a^2 - a + 2, 5a^2 - 4a - 7$.
 29. $-5x^2 + 3xy - y^2, -8x^2 + xy - y^2, 7x^2 - 3y^2$.
 30. $2x + 3y - 4z, -5x - y - z, -3x - 4y$.
 31. $b^2 - b^2c + c^2b, 2b^2c - 3c^2b, -4b^2 - 2c^2b$.
 32. $3a^2x^2 - 4ax - 10, 2a^2x^2 + ax - 6, -4a^2x^2 + ax + 9$.
 33. $a^2 - 5ab - 4b^2, -3a^2 + 2ab - 6b^2, 5a^2 + 3ab + 2b^2$.
 34. $5p^2 - 2pq + 7q^2, -9p^2 + 3pq - 6q^2, +2p^2 - 5pq + 10q^2$.
 35. $-1 - 3x - 2x^3, 5x - 3 + 4x^3, x^2 + 2x^3 - 2x + 2$.
 36. $3a - 4b + 5c - d + 6e, -2a + 4b - 2c - 7d + 3f,$
 $-6a + 4b - 3c - 7d + 9f, -5a - 2b + 5c - 4d - 3e,$
 $9a - 2b - c + 2e - 6f$.
 37. $3x^2 + 2y^2 + 6x - 9, 7x^2 - 3x - 4, -9x^2 - 7y^2 + 4,$
 $-4z^2 + yz - 3xy, 6y^2 + z^2 - 3xy - 3yz, 3z^2 + 4yz - 2xy$.
 38. $\frac{1}{2}a^2 - \frac{1}{3}ab + b^2, 2ab - \frac{1}{2}b^2 + \frac{3}{4}a^2, \frac{1}{3}a^2 + \frac{2}{3}ab - \frac{1}{3}b^2$.
 39. If $A = x^2 - 5x + 9, B = 4x - 3, C = 6 - x^2$, find the
 value of $A + B + C$.
 40. If $x = 4a^2 - 3a + 5, y = -2a + 7a^2 + 4$ and
 $z = -9 + 4a - 11a^2$, find the value of $x + y + z$.

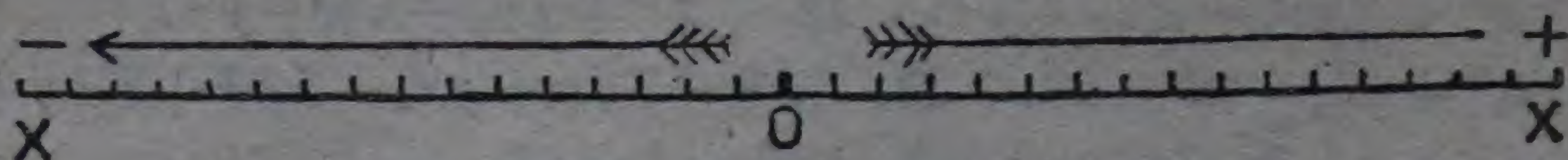
SUBTRACTION

5. *Principle.* In Arithmetic $8 - 5$ means that we have to find the number which must be added to 5 to make 8; or, *generally*, to find the number which must be added to the subtrahend to make the minuend.

In Algebra to subtract subtrahend from minuend means to find out:

- (i) How far the minuend is from the subtrahend, and
- (ii) the direction in which it lies.

Graphic Illustrations.



(i) To subtract $+5$ from $+8$ means marking the positions of $+5$ and $+8$ and finding out the *direction* and the *number* of steps to be taken from $+5$ to reach $+8$.

Evidently, in this case, we take three steps in the positive direction; hence the answer is $+3$.

(ii) To subtract -5 from $+8$ means marking the positions of -5 and $+8$ and finding out the *direction* and the *number* of steps to be taken from -5 to reach $+8$.

Evidently, in this case, we take thirteen steps in the positive direction; hence the answer is $+13$.

(iii) To subtract $+5$ from -8 means marking the positions of $+5$ and -8 and finding out the *direction* and the *number* of steps to be taken from $+5$ to reach -8 .

Evidently, in this case, we take thirteen steps in the negative direction; hence the answer is -13 .

(iv) To subtract -5 from -8 means marking the positions of -5 and -8 and finding out the *direction* and the *number* of steps to be taken from -5 to reach -8 .

Evidently, in this case, we take three steps in the negative direction; hence the answer is -3 .

Let us now compare algebraic addition with algebraic subtraction :

Algebraic addition		Algebraic subtraction	
(i) $+8$	(ii) $+8$	(i) $+8$	(ii) $+8$
-5	$+5$	$+5$	-5
<hr/>	<hr/>	<hr/>	<hr/>
$+3$	$+13$	$+3$	$+13$
(iii) -8	(iv) -8	(iii) -8	(iv) -8
-5	$+5$	$+5$	-5
<hr/>	<hr/>	<hr/>	<hr/>
-13	-3	-13	-3

DIRECTED NUMBERS, FIRST FOUR RULES

On comparison we find that the answer to

(i) in addition is the same as to	(i) in subtraction,
(ii) " " "	(ii) " "
(iii) " " "	(iii) " "
(iv) " " "	(iv) " "

but in each of the above corresponding cases the second terms (*i.e.*, the subtrahend and the corresponding term in addition) have opposite signs.

Rule. Change the sign of the subtrahend and then add.

When one compound expression is to be subtracted from another, we take the following steps:

- (i) Arrange the like terms in columns.
- (ii) Change the signs of all the terms in the subtrahend.
- (iii) Add the terms in columns.

Example 1. Subtract $3a - 5b - c$ from $5c - 8b + 4a$.

It means: To arrange the like terms in columns, change the signs of all the terms in the subtrahend, and then add.

Thus we have

$$\begin{array}{r}
 +4a - 8b + 5c \\
 -3a + 5b + c \\
 \hline
 + a - 3b + 6c
 \end{array}
 \begin{array}{l}
 \\
 \text{[Signs changed.]} \\
 \text{[Addition.]}
 \end{array}$$

Example 2. Subtract

$12xy - 5x^2 + 2c^2 - 1$ from $32x^2 + 20xy - 3y^2$.

Arranging the terms properly, changing the signs in the subtrahend, and adding the columns, we have

$$\begin{array}{r}
 32x^2 + 20xy - 3y^2 \\
 + 5x^2 - 12xy \quad - 2c^2 + 1 \\
 \hline
 37x^2 + 8xy - 3y^2 - 2c^2 + 1
 \end{array}$$

EXERCISE 10.

Subtract with the help of a graduated scale:

- | | |
|------------------------|-----------------------|
| 1. $+ 7$ from $+12$. | 2. $- 7$ from $+12$. |
| 3. $+ 7$ from $- 12$. | $- 7$ from -12 . |

5. $+12$ from $+7$.

6. -12 from $+7$.

7. $+12$ from -7 .

8. -12 from -7 .

Subtract (*orally*):-

9. $+7a$ from $+12a$.

10. $-7a$ from $+12a$.

11. $+7a$ from $-12a$.

12. $-7a$ from $-12a$.

13. $+12m$ from $+7m$.

14. $-12p$ from $+7p$.

15. $+12m$ from $-7m$.

16. $-12x$ from $-7x$.

17. $+5x$ from $-5x$.

18. $-7m$ from $+7m$.

19. $+9n$ from $+9n$.

20. $-13q$ from $-13q$.

21. $-m$ from 0.

22. $+m$ from 0.

23. $x-y$ from $x+y$.

24. $x-y$ from $x-y$.

25. $+b+c$ from a .

26. $-b+c$ from a .

27. $-3x+4y$ from $7x+5y$.

28. $+2m-6n$ from $3m-4n$.

Subtract by arranging like terms in columns:

29. $3x+2y+z$ from $5x+2y+3z$.

30. $2ab-3bc+7ca$ from $-ab+4bc+6ca$.

31. $-x^2+4y^2-xy$ from $-x^2-2y^2+3xy$.

32. $11-y^2+3x^2$ from $2x^2-3y^2-7$.

33. $1-m^2+m^3$ from $3-2m^2-m^4$.

34. $x^2+2y^2+z^2-6$ from $4x^2-7+z^2-2y^2$.

35. $a^2-2ab+3b^2$ from $-6a^2+ab-9b^2$.

36. $-8x^2y^2-3y^2z^2-z^2x^2$ from $-12x^2y^2-4y^2z^2+7z^2x^2$.

37. $3p^3+4p^2q-pq^2-2q^3+3$ from $4p^3-p^2q-pq^2+2q^3-3$.

38. $-x^2+\frac{1}{3}xy-4y^2+yz-z^2$ from $x^2+\frac{2}{3}xy-2y^2+yz-4z^2$.

39. Subtract a^2+b^2 from the sum of $a^2+2ab+b^2$ and $a^2-2ab+b^2$.

40. Subtract the sum of x^2+xy and $xy+y^2$ from the sum of x^2+xy+y^2 and x^2-xy+y^2 .

If $x=3a^2-1$, $y=2b^2+4$, $z=a^2-b^2+7$, find:

41. $x-y$.

42. $y-z$.

43. $z-x$.

44. $x+y-z$.

45. $x-y+z$.

46. $y+z-x$.

47. Which is greater and by how much:

(i) $-12, -7$?

(ii) $0, -4$?

6. Removal and Insertion of Brackets.

To add a and $+b+c$ is the same as $a + (+b+c)$, but we know that when a and $+b+c$ are added together, the sum is $a+b+c$.

$$\therefore a + (+b+c) = a + b + c.$$

Similarly, to add a and $-b+c$ is the same as $a + (-b+c)$, but we know that when a and $-b+c$ are added together, the sum is $a-b+c$.

$$\therefore a + (-b+c) = a - b + c.$$

Hence we conclude that **the brackets preceded by $+$ sign may be removed without making any change in the signs of the enclosed terms.**

To subtract $+b+c$ from a means the same as $a - (+b+c)$, but we know that when $+b+c$ is subtracted from a , the answer is $a-b-c$.

$$\therefore a - (+b+c) = a - b - c.$$

Similarly, to subtract $-b+c$ from a means the same as $a - (-b+c)$, but we know that when $-b+c$ is subtracted from a , the answer is $a+b-c$.

$$\therefore a - (-b+c) = a + b - c.$$

Hence we conclude that **the brackets preceded by $-$ sign may be removed on changing the signs of the enclosed terms, i.e., from $+$ to $-$ and from $-$ to $+$.**

Example 1. Simplify $7x + \{-3x + (5 - 2x)\}$.

$$\begin{aligned} \text{The expression} &= 7x + \{-3x + 5 - 2x\} \\ &= 7x + \{-5x + 5\} \\ &= 7x - 5x + 5 \\ &= 2x + 5. \end{aligned}$$

Example 2. Simplify $7x - \{-3x - (5 - 2x)\}$.

$$\begin{aligned} \text{The expression} &= 7x - \{-3x - 5 + 2x\} \\ &= 7x - \{-x - 5\} \\ &= 7x + x + 5 \\ &= 8x + 5. \end{aligned}$$

Example 3. Simplify by removing the brackets

$$(a + b + c + d) - (a - b + c - d) + (a - b + c - d).$$

$$\begin{aligned}\text{The expression} &= a + b + c + d - a + b - c + d + a - b + c - d \\ &= a + b + c + d.\end{aligned}$$

Example 4. Shew that

$$(1 + 3x) - (2 - 5x) + (2 - 3x) - (1 + 5x) = 0.$$

$$\begin{aligned}\text{The expression} &= 1 + 3x - 2 + 5x + 2 - 3x - 1 - 5x \\ &= 1 - 2 + 2 - 1 + 3x + 5x - 3x - 5x \\ &= 1 + 2 - 2 - 1 + 3x + 5x - 3x - 5x \\ &= 3 - 3 + 8x - 8x = 0.\end{aligned}$$

EXERCISE 11.

Write down the value of the following without any process:

1. (i) $+(+5)$, (ii) $-(+3)$, (iii) $+(-7)$, (iv) $-(-6)$.
2. (i) $-(-3m)$, (ii) $+(-5q)$, (iii) $-2-(2-2)$,
(iv) $-(2-4)-2$.
3. (i) $-x^2 - (-a^2) + (-a^2) + (+x^2)$, (ii) $(+p-p) - (p-p)$.
4. (i) $+(-1) + (-1) + (-1)$, (ii) $-(+3) - (+3) - (+3)$,
(iii) $-(-3) - (-3) - (-3)$.

Simplify :

5. $4p + (-3q - 5p)$.
6. $(4a - 5b) + (-2a + 4b)$.
7. $(6p - 2q) + (3q - 4p)$.
8. $4p - (-3q + 5p)$.
9. $7k - (-4k - 2l)$.
10. $(3a - 4b) - (-2a + 5b)$.
11. $(l - 5m + 2n) - (-3n + 4m - 5l)$.
12. $-3a - \overline{3b + c} + \overline{3a + 3b} - (+c)$.
13. $p + \{ 2q + \overline{3 - 5p} - (-q + 7) + 5 \}$.
14. $2p - (-3 + \overline{p - 2q - 5}) - 7$.
15. $2a + (+5b - \overline{7b - 2a})$.
16. $2 - \{ a - 3 + \overline{-5} \}$.
17. $5y - [3x + \{ 2y - (3y - z) + 5z \}]$.
18. $-(-(-(-(-(-1)))))$.
19. $-3 + (-(+(-(+(-1)))))$.

20. $1 - [1 - \{1 - (1 - \overline{1 - 1})\}].$

21. If $a = -1$, $b = -2$, $c = -3$, find the value of
 $1 - b + a - [c - \{a + (b - c - \overline{a - b - c})\}].$

Find the value of :

22. $(a + b + c) - (a - b - c) + (-a - b - c) + (a + b - c).$

23. $(x + y - z) + (y + z - x) - (z + x - y) - (x + y + z).$

24. $(a - b + c) - (3a - 3b + 3c) + (2a - 2b + 2c).$

25. $(a^3 + b^3 - c^3 + abc) - (a^3 - b^3 + c^3 - abc) +$
 $(a^3 - b^3 + c^3 - abc) - (a^3 + b^3 - c^3 + abc).$

26. $[a - \{b + (c + d)\}] - [a + \{b - (c - d)\}]$
 $+ [a - \{b - (c + d)\}] - [a - \{b - (c - d)\}].$

27. Find whether

$5 - (3 - 9)$ or $9 - (5 - 3)$ is greater.

From the above rules for the removal of brackets, we deduce the following rules for the **insertion of brackets** :

(i) Any number of terms in an expression may be put within a pair of brackets and the sign $+$ prefixed.

(ii) Any number of terms in an expression may be put within a pair of brackets and the sign $-$ prefixed, provided the signs of all the terms placed within the brackets be changed, i.e., from $+$ to $-$ and from $-$ to $+$.

For example, $a + b - c + d - e + f$ may be written in the form

$a + b - (c - d + e - f)$

or $a - (-b + c - d + e - f)$

or $+(a + b) - (c - d) - (e - f)$

or $-(-a - b + c) + (d - e + f)$

or $-(-a - b) + (-c + d) - (e - f).$

28. Enclose all the terms after the first, within a pair of brackets preceded by (a) positive sign, (b) negative sign:

(i) $-a^2 + m^2 + b^2 - n^2 + b^2 + c^2,$

(ii) $x^2 - a^2 + bc - b^2 - ca + c^2 - ab.$

* 29. Bracket like powers of x , so that the sign before the brackets may be (a) positive, (b) negative :

(i) $2x^4 + 3x^2 - 4x^3 + x^2 - 5x^4 + x^3$

(ii) $ax^3 - bx^2 + cx - dx^3 - ex^2 - fx$.

* 30. Fill in the blanks :

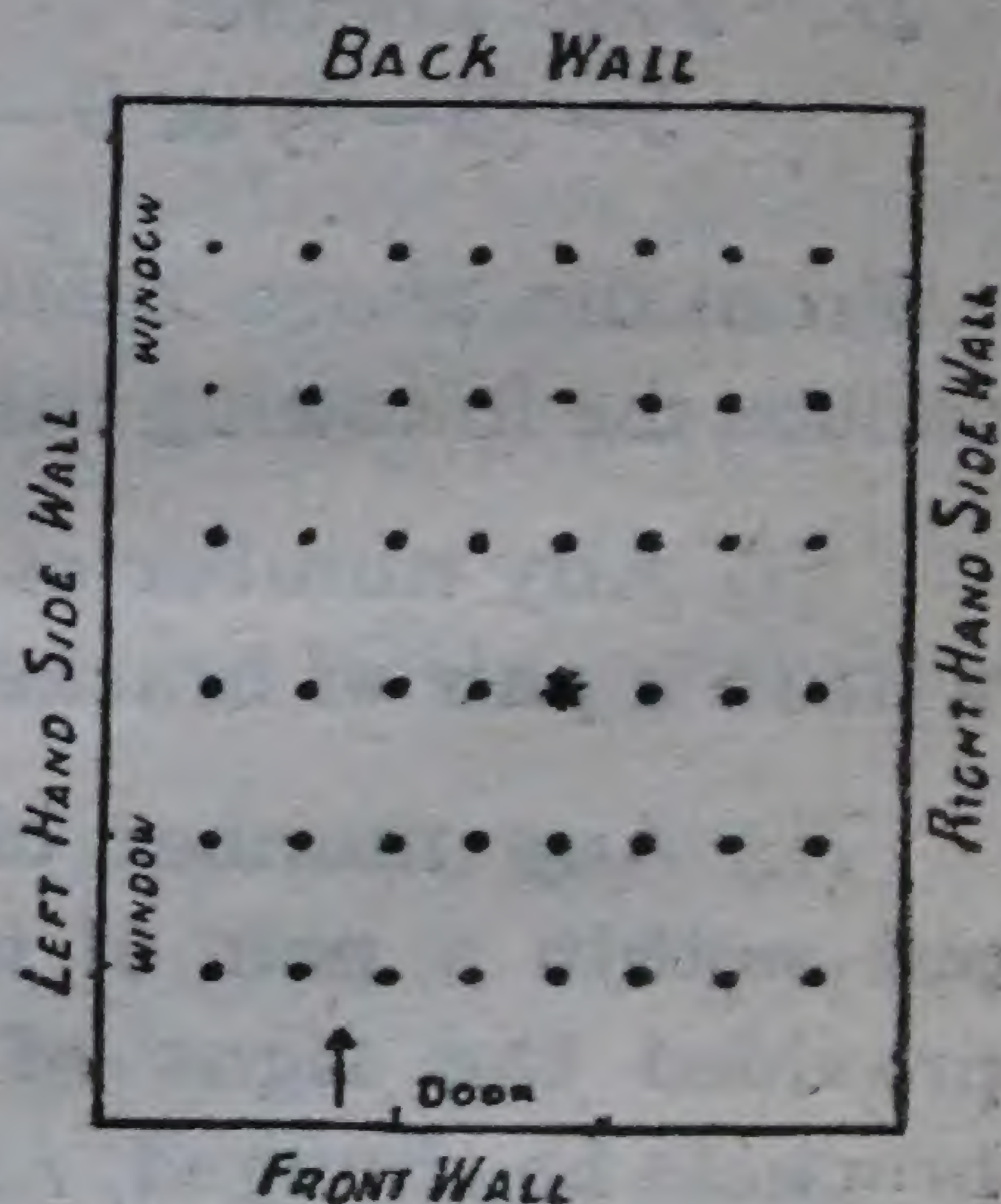
(i) $-1 = +(-(+(-(+ (\quad)))))$,

(ii) $+a = -(-(-(+(-(+ (\quad))))))$.

7. Co-ordinates of a point. The annexed figure is the plan of an examination hall. The position of each candidate is indicated by means of a dot.

Suppose the position * of a particular candidate is to be pointed out to the superintendent from some distance.

If we merely say that he is in the 3rd row, it is not enough, for the 3rd row may be from the front wall or back wall. Again, if we say that he is in the 3rd row from the front wall, even that is not enough, for there are in all 8 candidates in that row and he may be *any* out of these 8.

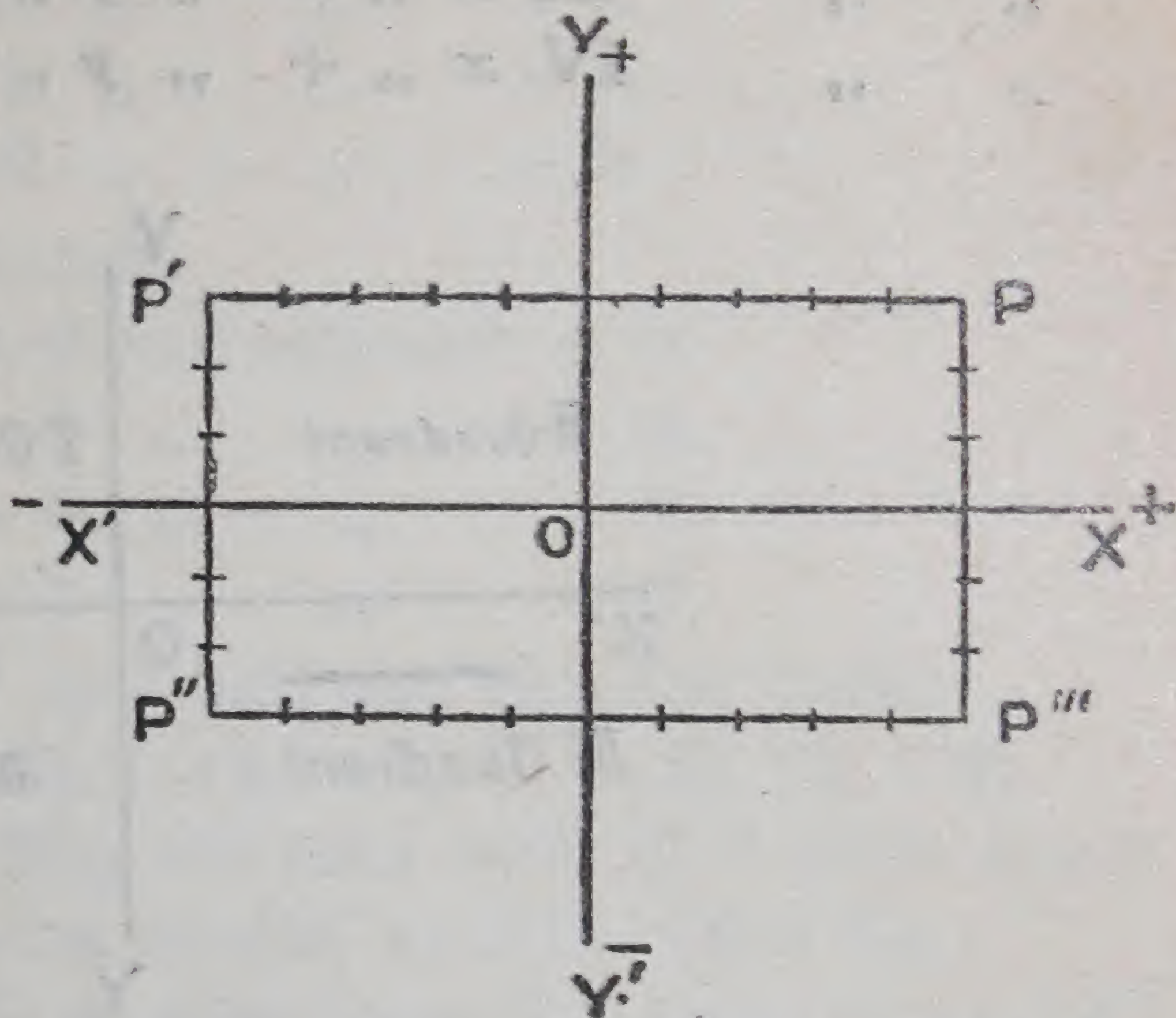


Even if we add that he is in the 5th column, it is not enough, for the 5th column may be from the left-hand side wall or from the right-hand side wall. If, however, we say that he is in the 5th column from the left-hand side wall, his position is definitely fixed.

Hence to locate definitely the position of a candidate in an examination hall, we have to refer to his position with respect to two adjacent walls.

DIRECTED NUMBERS, FIRST FOUR RULES

If we have to fix the position of a point P in a plane, with reference to two straight lines (say XOX' and YOY') at right angles to one another, it is not enough to say that the point P is 5' from YOY' and 3' from XOX' , for there could be 4 points P, P', P'', P''' (see fig.) each satisfying the given condition; but if we say that it is at the distance of + 5' from YOY' and + 3' from XOX' , the point is definitely fixed in position.



In order to fix the position of P' , we shall have to say that it is - 5' from YOY' and + 3' from XOX' .

Similarly, the point P'' is - 5' from YOY' and - 3' from XOX' , and the point P''' is + 5' from YOY' and - 3' from XOX' .

These distances of a point from the lines of reference are called **co-ordinates** of the point.

Thus, the co-ordinates of P are (+ 5, + 3),

“ “ “ P' “ (- 5, + 3),

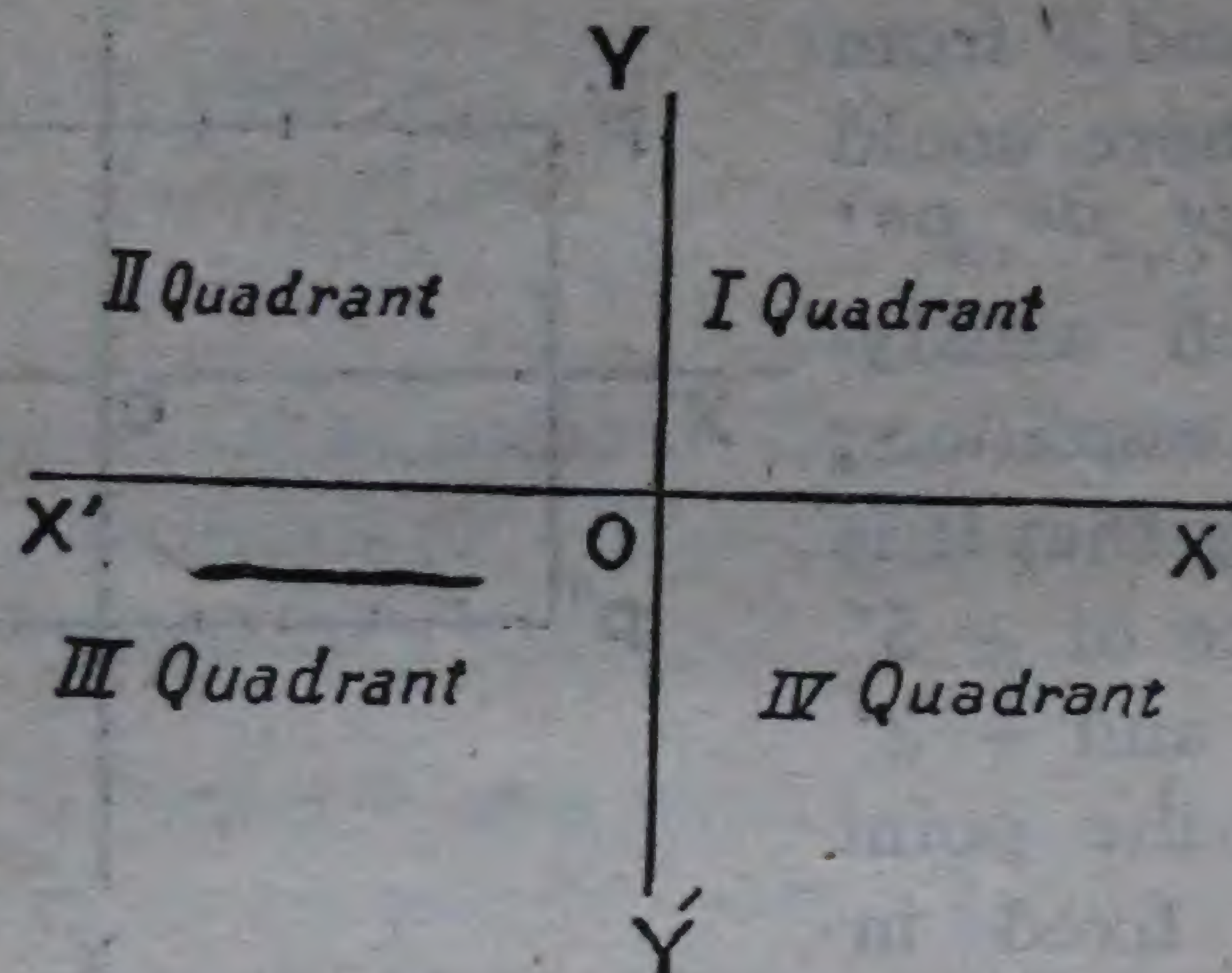
“ “ “ P'' “ (- 5, - 3),

“ “ “ P''' “ (+ 5, - 3).

The distance of a point from YOY' is called the **abscissa** and is usually represented by x , whereas its distance from XOX' is called the **ordinate** and is usually represented by y . The lines XOX' and YOY' are called the x -axis and y -axis respectively. O , the point of intersection of the axes from which all measurements are made, is called the **origin**.

The plane is divided into 4 compartments by the axes, and each of them is called a **quadrant**. (See figure on page 48.)

In quadrant	I	x is +	and y is +
"	II	x ,, -	" y ,, +
"	III	x ,, -	" y ,, -
"	IV	x ,, +	" y ,, -.

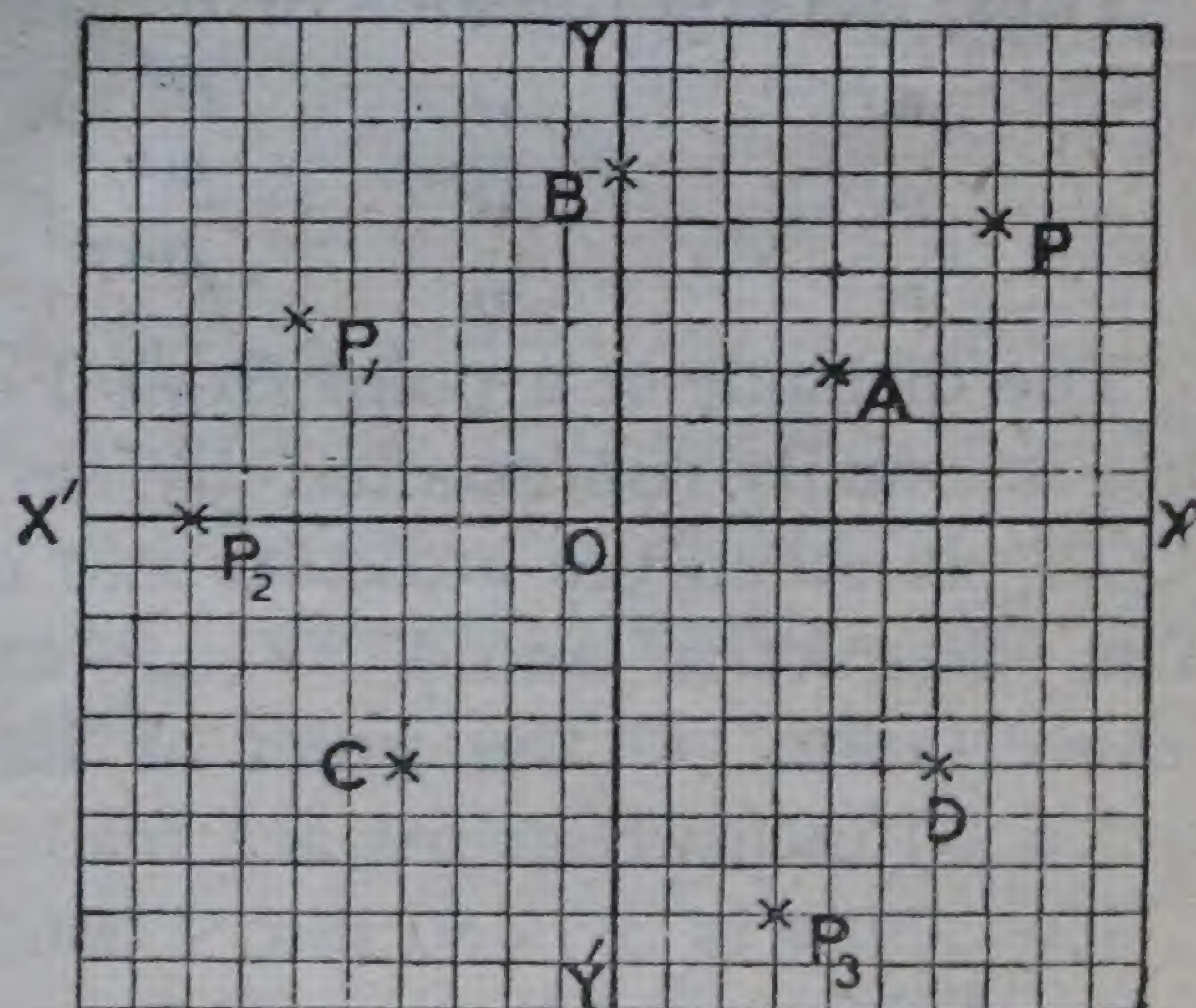


The process of determining the position of a point in a plane, with reference to the two axes, is called *plotting the point*.

For the graphic representation of points and lines we use ordinary squared paper, in which the side of a small square is $\cdot 1''$ and the side of a big square is $1''$.

Example 1. (i) Write down the co-ordinates of the points A, B, C, D taking $\cdot 1''$ as a unit.

(ii) Write down the co-ordinates of the points P, P_1, P_2, P_3 , taking $1''$ as a unit. (See fig.)



DIRECTED NUMBERS, FIRST FOUR RULES

(i) If we take $\cdot 1''$ as a unit,
the abscissa of A is $+4$ and ordinate $+3$,

„ „ „ B „ 0 „ „ $+7$,

„ „ „ C „ -4 „ „ -5 .

„ „ „ D „ $+6$ „ „ -5 .

(ii) If we take $1''$ as a unit,
the abscissa of P is $+\cdot 7$ and ordinate $+\cdot 6$,

„ „ „ P_1 „ $-\cdot 6$ „ „ $+\cdot 4$,

„ „ „ P_2 „ $-\cdot 8$ „ „ 0 ,

„ „ „ P_3 „ $+\cdot 3$ „ „ $-\cdot 8$.

Example 2. Take $\cdot 1''$ as a unit and plot the points

$(7, 10)$, $(-5, 6)$, $(-8, 0)$, $(-4, -5)$, $(0, -6)$, $(3, -8)$.

To plot the point $(7, 10)$, we take from O , 7 units along OX and from there 10 units parallel to OY . (See fig.)

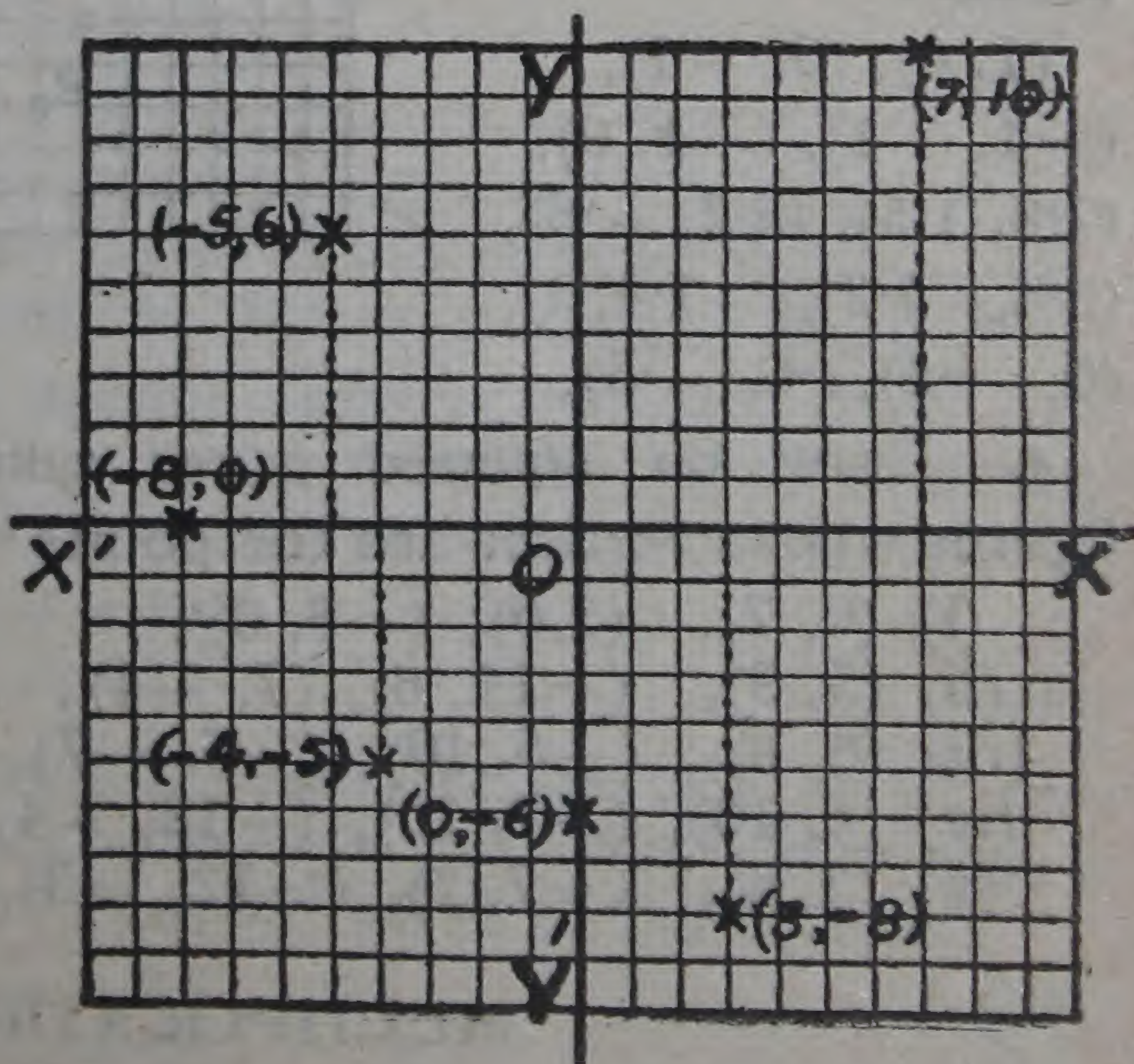
To plot the point $(-5, 6)$ we take from O , 5 units along OX' and from there 6 units parallel to OY . (See fig.)

To plot the point $(-8, 0)$, we take from O , 8 units along OX' and from there no unit above or below it. (See fig.)

To plot the point $(-4, -5)$, we take from O , 4 units along OX' and from there 5 units parallel to OY' . (See fig.)

To plot the point $(0, -6)$, we take no units along OX or OX' and 6 units parallel to OY' from O . (See fig.)

To plot the point $(3, -8)$, we take from O , 3 units along OX and from there 8 units parallel to OY' . (See fig.)



EXERCISE 12

1. Write down the co-ordinates of the points $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9$ in this figure :

(i) taking $\cdot 1''$ as a unit.

(ii) taking $1''$ as a unit.

2. Take $\cdot 1''$ as a unit and plot the points:

$(9, 7), (0, 6), (-5, 8),$

$(6, 0), (-7, -4).$

$(0, 7), (-8, -5),$

$(-7, +8), (9, -6).$

$(17, -1), (-20, 4),$

$(0, 0), (13, 0), (0, -13),$

$(9, -9), (-9, -9),$

$(0, 11).$

3. Take $1''$ as a unit and plot the points :

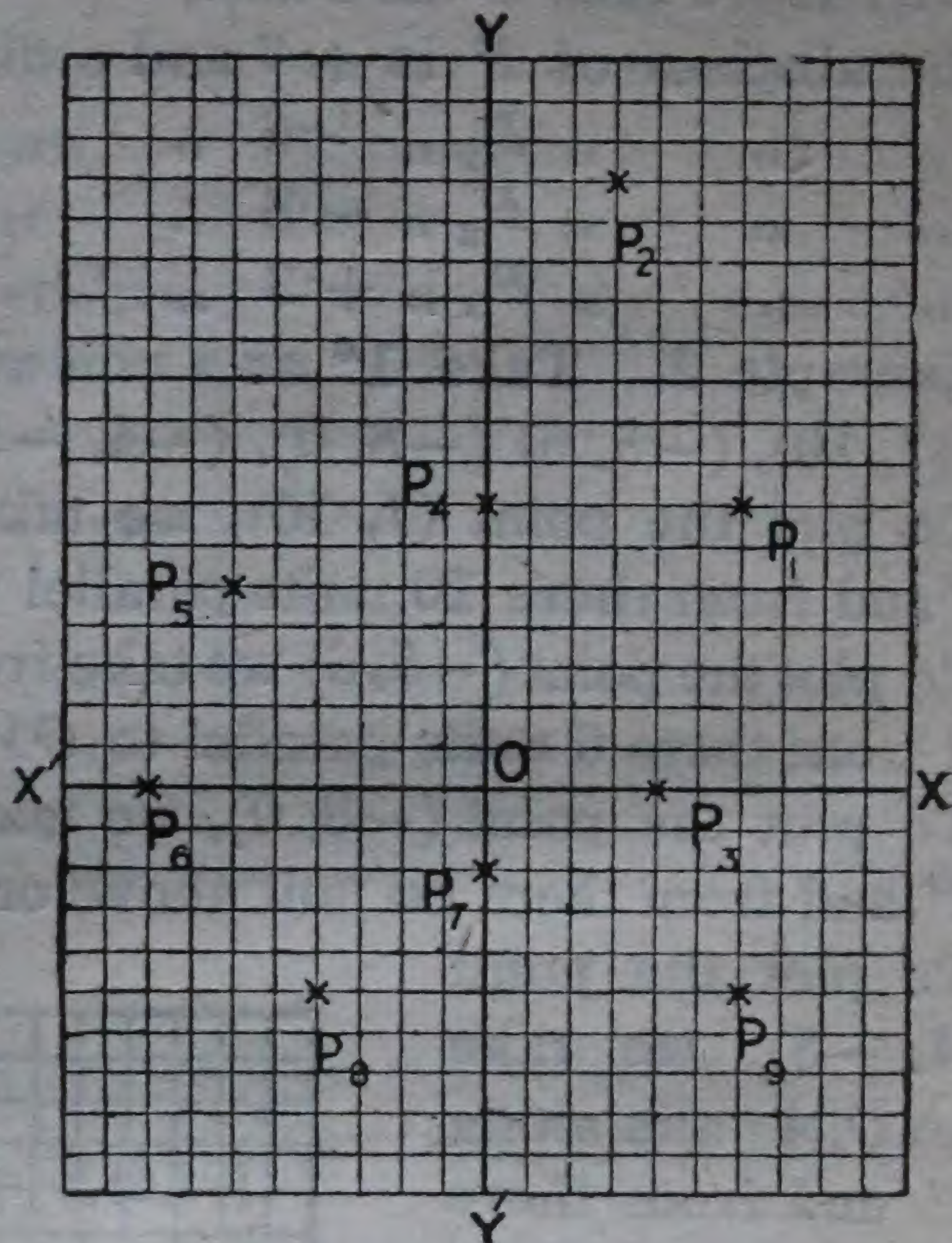
$(1, 0), (0, -1),$

$(-1, -1), (-1, 0),$

$(1\cdot3, 1\cdot5), (3\cdot2, -\cdot6),$

$(3\cdot6, -1\cdot2), (-1\cdot4, 0),$

$(0, -\cdot5), (\cdot6, -\cdot5).$



4. Draw on squared paper taking $\cdot 1''$ as a unit, the figures whose vertices are the points :

(i) $(0, 7), (5, 0), (-5, 0);$

(ii) $(5, 9), (-11, 6), (7, -4);$

(iii) $(6, 8), (-6, 10), (-5, -7), (12, -9);$

(iv) $(6, 10), (-7, 10), (-12, -4), (14, -4);$

(v) $(3, 6), (-7, 7), (-12, -3), (-4, -6), (9, -8).$

MULTIPLICATION

8. Law of signs. (a) Let us consider the changes in the annual saving of a municipality by the arrival and

departure of taxpayers and orphans, the former paying and the latter receiving Rs. 8 each annually.

(i) If five new taxpayers come in, the municipal saving is increased by Rs. 40.

(ii) If five taxpayers go away, the saving is decreased by Rs. 40.

(iii) If five new orphans come in, the saving is decreased by Rs. 40.

(iv) If five orphans go away, the saving is increased by Rs. 40.

If a tax of Rs. 8 be represented by $+8$, an increase of Rs. 40 in the saving would be represented by $+40$; a payment of Rs. 8 would be represented by -8 , and a decrease of Rs. 40 in the saving would be represented by -40 . If the coming of five men be represented by $+5$, the going away of five men would be represented by -5 .

With such notation, the above four processes stand thus :

$$\begin{array}{llll} (+8) \times (+5) = +40 & \cdot & \cdot & \cdot & (i) \\ (+8) \times (-5) = -40 & \cdot & \cdot & \cdot & (ii) \\ (-8) \times (+5) = -40 & \cdot & \cdot & \cdot & (iii) \\ (-8) \times (-5) = +40 & \cdot & \cdot & \cdot & (iv) \end{array}$$

(b) Let us consider the position of a ship, taking the equator as the starting place, the degrees north of the equator positive, the degrees south of the equator negative, the days to come positive, and the days passed away negative.

(i) Suppose a ship sailing *north* at the rate of 3° per day is at the equator, 5 days hence it will be 15° *north* of the equator, and 5 days back it was 15° *south* of the equator :

$$\begin{array}{llll} \text{or,} & (+3) \times (+5) = +15 & \cdot & \cdot & \cdot & (i) \\ & (+3) \times (-5) = -15 & \cdot & \cdot & \cdot & (ii) \end{array}$$

(ii) Suppose a ship sailing *south* at the rate of 3° per day is at the equator, 5 days hence it will be 15° *south* of the equator and 5 days back it was 15° *north* of the equator :

$$\begin{array}{llll} \text{or,} & (-3) \times (+5) = -15 & \cdot & \cdot & \cdot & (iii) \\ & (-3) \times (-5) = +15 & \cdot & \cdot & \cdot & (iv) \end{array}$$

From these and similar examples, we generalise the process as given on the next page :

MATRICULATION ALGEBRA

$$(+a) \times (+b) = +ab \quad . \quad . \quad . \quad (i)$$

$$(-a) \times (+b) = -ab \quad . \quad . \quad . \quad (ii)$$

$$(+a) \times (-b) = -ab \quad . \quad . \quad . \quad (iii)$$

$$(-a) \times (-b) = +ab \quad . \quad . \quad . \quad (iv)$$

Law. The product of two quantities with like signs is positive.

The product of two quantities with unlike signs is negative.

NOTE. (i) $(+a) \times 0 = +(a \times 0) = +0 = 0$,

(ii) $0 \times (+a) = +(0 \times a) = +0 = 0$,

(iii) $(-a) \times 0 = -(a \times 0) = -0 = 0$,

(iv) $0 \times (-a) = -(0 \times a) = -0 = 0$.

Hence, if one factor is zero the product is zero.

Graphic Illustrations

Convention. An area is usually considered positive, if it is on our left-hand side, when we go round it. (See fig. 1.)

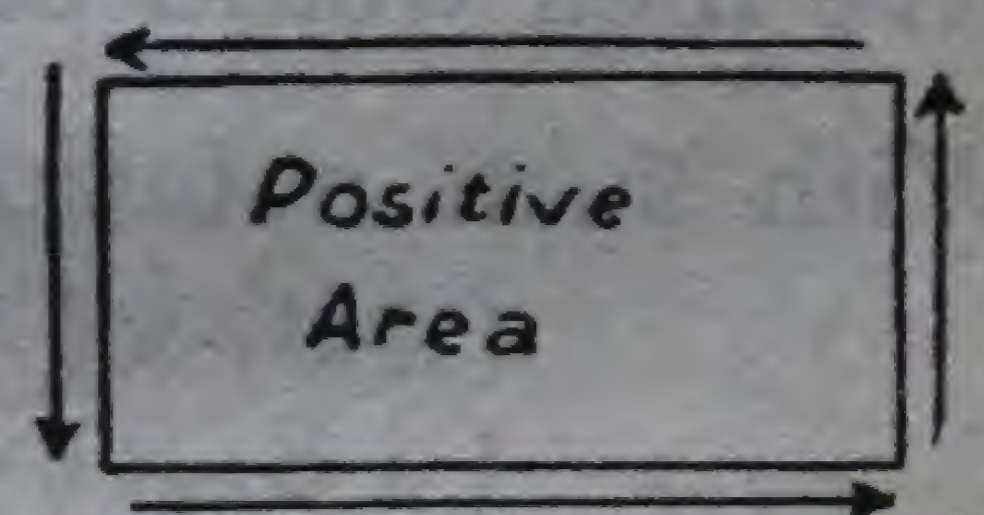


Fig. 1.

An area is usually considered negative, if it is on our right-hand side when we go round it. (See fig. 2.)

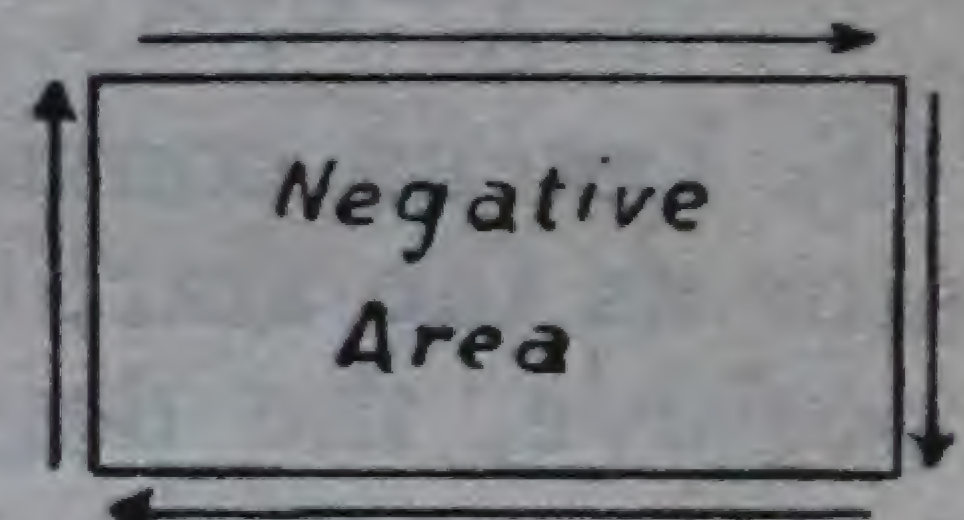
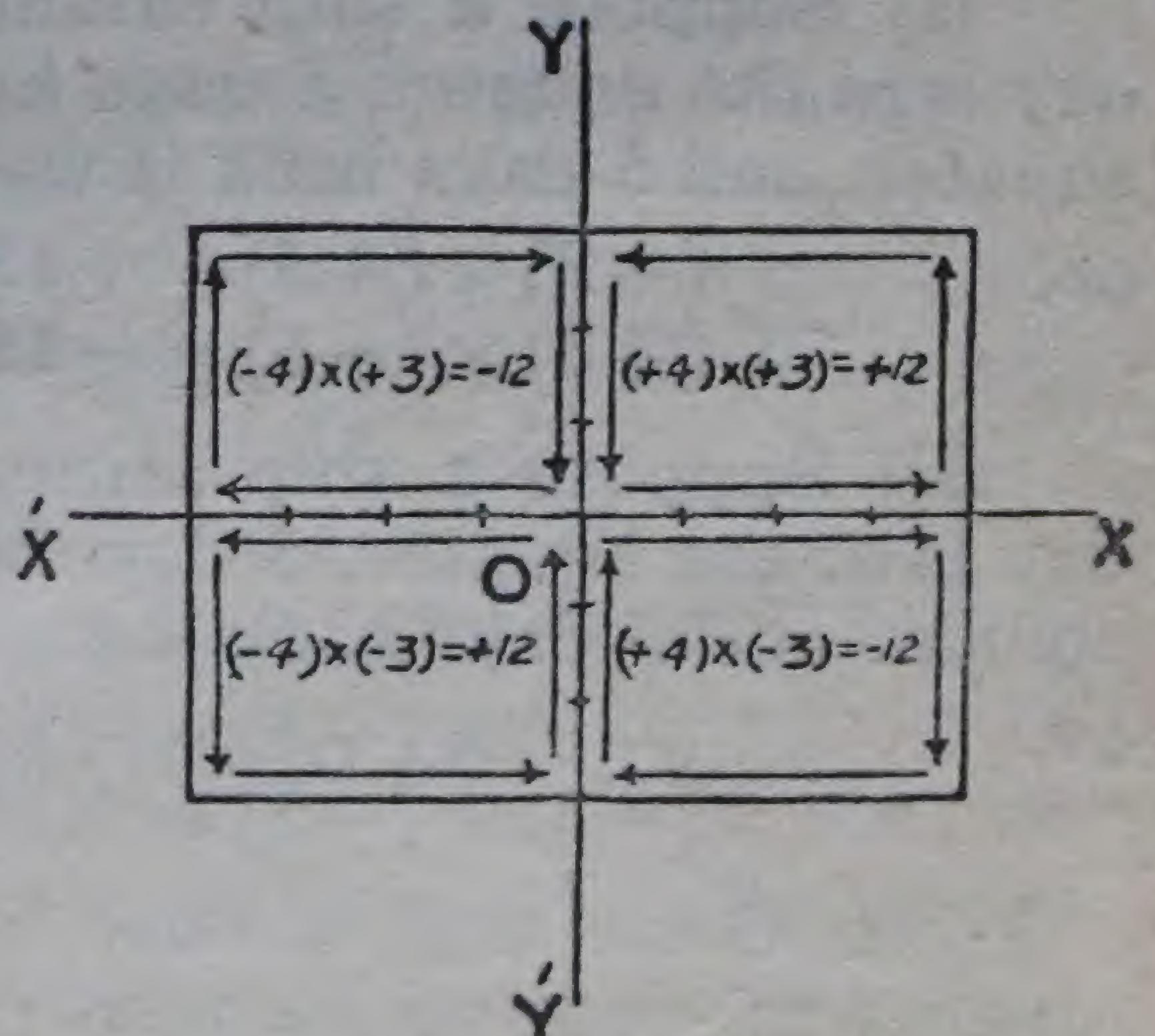


Fig. 2.

With this convention, we can easily illustrate the law of sign *graphically*, as discussed below:

XX' and YY' are the axes and O is the origin. In each quadrant a rectangle is constructed whose sides are 4 units and 3 units. In each case



we start from the origin, move along the x -axis and then complete the circuit. It is obvious that

(i) the sides of the rectangle in the first quadrant are $(+4)$ and $(+3)$, and its area is $(+12)$;

(ii) the sides of the rectangle in the second quadrant are (-4) and $(+3)$, and its area is (-12) ;

(iii) the sides of the rectangle in the third quadrant are (-4) and (-3) , and its area is $(+12)$;

(iv) the sides of the rectangle in the fourth quadrant are $(+4)$ and (-3) , and its area is (-12) .

EXERCISE 13.

1. Multiply (*orally*):

- | | |
|--------------------------|-------------------------|
| (i) $(+5)$ by $(+7)$, | (ii) $(+7)$ by (-2) , |
| (iii) (-9) by $(+5)$, | (iv) (-8) by (-4) , |
| (v) 12 by -5 , | (vi) -15 by 8 , |
| (vii) $+5a$ by $+8$, | (viii) $+7m$ by -4 , |
| (ix) $-8m$ by $+4$, | (x) $-7m$ by -8 , |
| (xi) $6a$ by -12 , | (xii) 12 by $-2m$, |
| (xiii) -8 by 0 , | (xiv) 0 by -6 |
| | (xv) $+3a$ by 0 . |

2. Multiply (*orally*):

- | | |
|--|---|
| (i) $4 \times 0 \times (-3)$, | (ii) $(+4m) \times 0 \times (-3m)$, |
| (iii) $(+2a) \times (-3b) \times 0$, | (iv) $0 \times (-7p) \times (-2q)$, |
| (v) $(+5) \times (+5)$, | (vi) $(-5) \times (-5)$, |
| (vii) $(-a) \times (-a)$, | (viii) $(-b) \times (-b) \times (-b)$, |
| (ix) $(-c) \times (-c) \times (-c) \times (-c)$, | |
| (x) $(-p) \times (-p) \times (-p) \times (-p) \times (-p)$, | |
| (xi) $(-1) \times (-1)$, | (xii) $(-1) \times (-1) \times (-1)$, |
| (xiii) $(-1) \times (-1) \times (-1) \times (-1)$, | |
| (xiv) $(-1) \times (-1) \times (-1) \times (-1) \times (-1)$. | |

3. Show that:

- (i) $(+a)^2 = +a^2$ and $(-a)^2 = +a^2$,
- (ii) $(+x)^3 = +x^3$ and $(-x)^3 = -x^3$,
- (iii) $(+p)^4 = +p^4$ and $(-p)^4 = +p^4$,

$$(iv) (+m)^5 = +m^5 \text{ and } (-m)^5 = -m^5,$$

$$(v) (+1)^2 = +1 \text{ and } (-1)^2 = +1,$$

$$(vi) (+1)^3 = +1 \text{ and } (-1)^3 = -1,$$

$$(vii) (+1)^4 = +1 \text{ and } (-1)^4 = +1,$$

$$(viii) (+1)^5 = +1 \text{ and } (-1)^5 = -1.$$

4. What rule do you deduce from the above eight examples? Apply that rule to find out the values of the following :

$$(i) (-1)^{57}, (ii) (-1)^{64}, (iii) (-a)^{83}, (iv) (-a)^{94},$$

$$(v) (+p)^{43}, (vi) (+m)^{54}, (vii) (+n)^{71}.$$

5. $8x=0$, what is the value of x ?

6. $4(x-3)=0$, what is the value of $x-3$?

7. Find the value of $3x^3 - 2x^2y - 5xy^2 + y^3$ by substituting for x and y the values given in the table :

x	-3	-2	$+1$	-1	-1	0
y	$+2$	-3	-2	-1	0	-1

8. Find the value of :

$$(i) 5x^3 - 3x^2 + 4x + 1,$$

$$(ii) -3x^4 + x^3 + 2x^2 - x + 4,$$

$$(iii) -4x^3 - 2x^2 + 6x - 9,$$

when x has values $-2, +3, -1, 0$.

9. Find the product of :

$$(i) (-2y)^3, (3y)^4, (-y)^5,$$

$$(ii) (-a)^6, (-2ab^2)^2, (-a^2b)^3.$$

$$10. \text{ Simplify } 4x^2(x^2 - 3xy + 2y^2) - 2xy(3x^2 - 2xy + y^2) - y^2(2x^2 - xy - y^2).$$

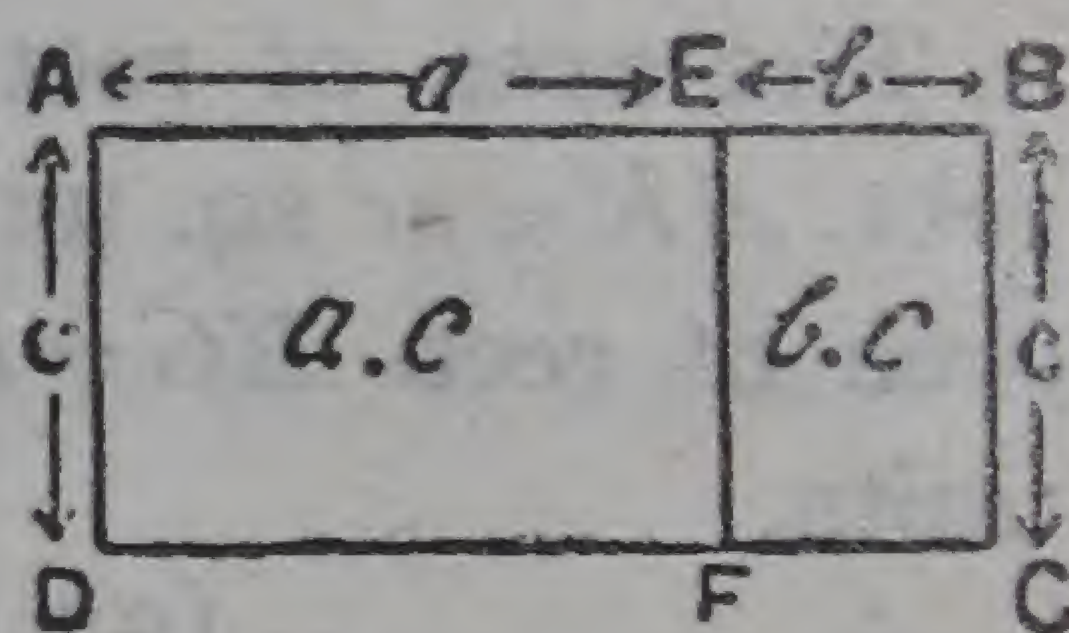
9. Just as $(a+b) \times 2 = (a+b) + (a+b)$
 $= a+b+a+b = 2a+2b,$
 and $(a-b) \times 2 = (a-b) + (a-b)$
 $= a-b+a-b = 2a-2b,$
 so $(a+b) \times c = ac+bc,$
 and $(a-b) \times c = ac-bc.$

Graphic Illustrations. (i) Take a rectangle $ABCD$.

$AB = (a+b)$ units, $AD = c$ units,
 $AE = a$ units, and $EB = b$ units.

Draw EF parallel to AD .

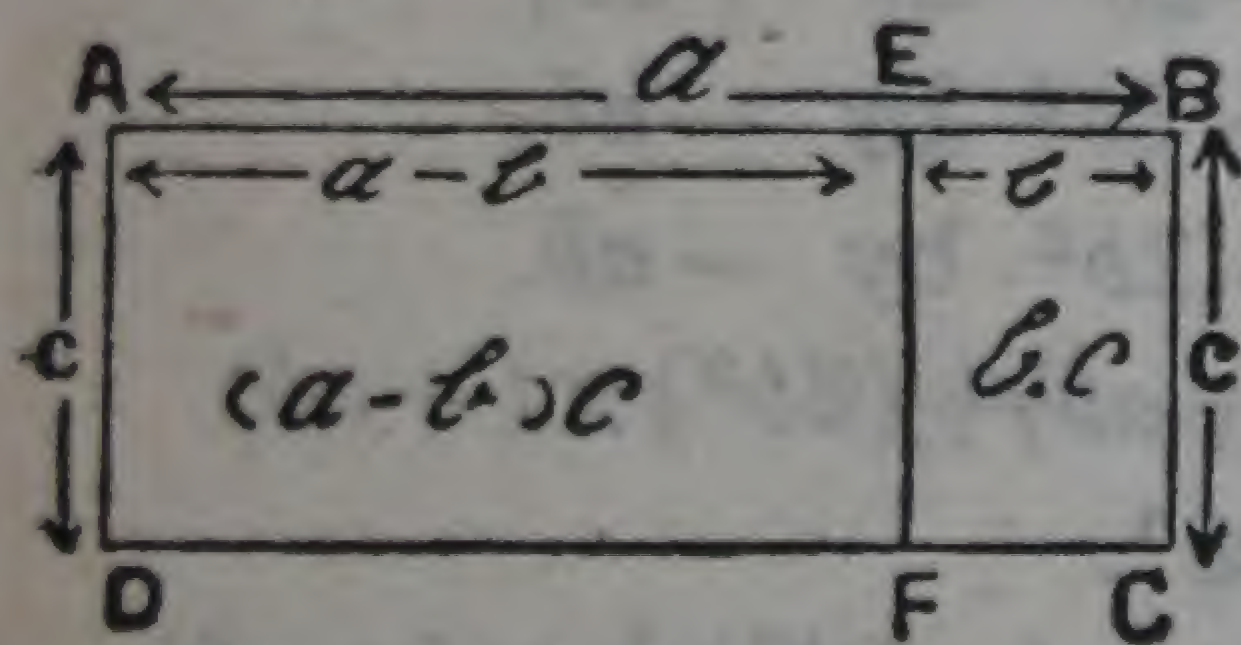
The whole rectangle AC
 $= \text{rect. } AF + \text{rect. } EC.$



\therefore The area of rectangle $AC = (a+b)c$ sq. units, the area of rectangle $AF = ac$ sq. units and that of rectangle $EC = bc$ sq. units,

$$\therefore (a+b)c = ac + bc.$$

(ii) Take a rectangle $ABCD$.



$AB = a$ units, $EB = b$ units,
 and $AD = c$ units.

$$\therefore AE = (a-b) \text{ units,}$$

The rect. $AF = \text{rect. } AC - \text{rect. } EC.$

\therefore The area of rect. $AF = (a-b)c$ sq. units, the area of rect. $AC = ac$ sq. units, and that of rect. $EC = bc$ sq. units,

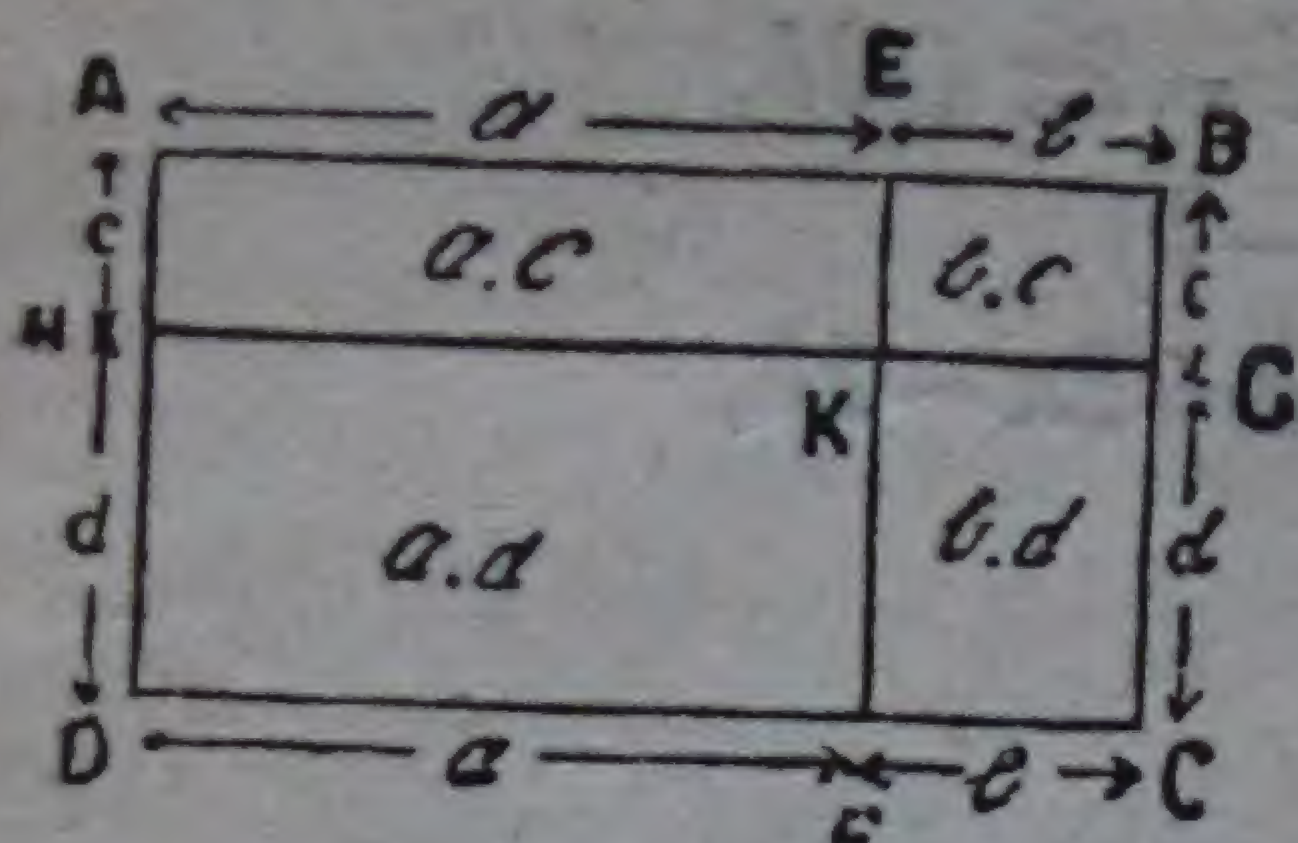
$$\therefore (a-b)c = ac - bc.$$

Rule. The product of a compound expression and a single factor is found by multiplying in succession each term of the expression by that factor.

Since $(a+b)m = am + bm,$
 if we substitute $c+d$ for m , we have

$$\begin{aligned} (a+b)(c+d) &= a(c+d) + b(c+d) \\ &= ac + ad + bc + bd. \end{aligned}$$

Graphic Illustration. Take a rectangle $ABCD$ in which



$AE = a$ units, $EB = b$ units,
 $AH = c$ units, and $HD = d$ units.

Through E and H draw straight lines parallel to the opposite sides, intersecting at K .

Rect. $AC = \text{rect. } AK + \text{rect. } HF + \text{rect. } EG + \text{rect. } KC$.

The area of rect. $AC = (a + b)(c + d)$ sq. units, the area of rect. $AK = ac$ sq. units, the area of rect. $HF = ad$ sq. units, the area of rect. $EG = bc$ sq. units, and that of rect. $KC = bd$ sq. units.

$$\therefore (a + b)(c + d) = ac + ad + bc + bd.$$

Rule. Multiply each term of the first expression by each term of the second and add all the partial products thus obtained.

Example 1. Multiply (i) $3a^2 + 2b^2$ by $+ab$,

(ii) $3a^2 - 2b^2$ by $+ab$,

(iii) $3a^2 + 2b^2$ by $-ab$,

(iv) $3a^2 - 2b^2$ by $-ab$.

$$\begin{aligned} \text{(i)} \quad (3a^2 + 2b^2) \times (+ab) &= (3a^2) \times (+ab) + (2b^2) \times (+ab) \\ &= 3a^3b + 2ab^3. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad (3a^2 - 2b^2) \times (+ab) &= \{ (+3a^2) + (-2b^2) \} \times (+ab) \\ &= (+3a^2) \times (+ab) + (-2b^2) \times (+ab) \\ &= 3a^3b - 2ab^3. \end{aligned}$$

$$\begin{aligned} \text{(iii)} \quad (3a^2 + 2b^2) \times (-ab) &= (3a^2) \times (-ab) + (+2b^2) \times (-ab) \\ &= -3a^3b - 2ab^3. \end{aligned}$$

$$\begin{aligned} \text{(iv)} \quad (3a^2 - 2b^2) \times (-ab) &= \{ (+3a^2) + (-2b^2) \} \times (-ab) \\ &= (+3a^2) \times (-ab) + (-2b^2) \times (-ab) \\ &= -3a^3b + 2ab^3 \end{aligned}$$

Example 2. Multiply (i) $(3x + 2y)$ by $(3x + 4y)$,

(ii) $(3x + 2y)$ by $(3x - 4y)$,

(iii) $(3x - 2y)$ by $(3x - 4y)$.

$$\begin{aligned}
 \text{(i)} \quad (3x+2y) \times (3x+4y) &= (3x+2y) (+3x) + (3x+2y) (+4y) \\
 &= 9x^2 + 6xy + 12xy + 8y^2 \\
 &= 9x^2 + 18xy + 8y^2.
 \end{aligned}$$

The process is usually arranged thus :

$$\begin{array}{r}
 3x+2y \\
 3x+4y \\
 \hline
 (3x+2y) \times (+3x) = 9x^2 + 6xy \\
 (3x+2y) \times (+4y) = \quad + 12xy + 8y^2 \\
 \hline
 \text{Product} \quad = 9x^2 + 18xy + 8y^2
 \end{array}$$

After some practice, each partial product is written with like terms in columns and the left-hand side of the process is omitted, as illustrated below :

$$\begin{array}{r}
 \text{(ii)} \quad \begin{array}{r} 3x+2y \\ 3x-4y \\ \hline 9x^2+6xy \\ -12xy-8y^2 \\ \hline 9x^2-6xy-8y^2 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \text{(iii)} \quad \begin{array}{r} 3x-2y \\ 3x-4y \\ \hline 9x^2-6xy \\ -12xy+8y^2 \\ \hline 9x^2-18xy+8y^2 \end{array}
 \end{array}$$

EXERCISE 14.

1. Multiply (*orally*):

- | | |
|---------------------------------|----------------------------------|
| (i) $(4a+5b) \times (+3),$ | (ii) $(5x+7y) \times (-3),$ |
| (iii) $(2m-3n) \times (+7),$ | (iv) $(3m-5n) \times (-7),$ |
| (v) $(3p-2q) \times (+4p),$ | (vi) $(3p-2q) \times (-5q),$ |
| (vii) $(2a+b-3c) \times (+4c),$ | (viii) $(2a-b-2c) \times (-5c).$ |

Multiply :

- | | |
|------------------------------------|---------------------------------|
| 2. $(x+2) \times (x+3).$ | 3. $(x+5) \times (x+2).$ |
| 4. $(a+b) \times (b+c).$ | 5. $(a+b) \times (a+b).$ |
| 6. $(2a+b) \times (3a+2b).$ | 7. $(2x+3y) \times (x+y).$ |
| 8. $(a^2-ab) \times (ab-b^2).$ | 9. $(x^3-10) \times (-x^3+10).$ |
| 10. $(-x^2+2xy) \times (3xy-y^2).$ | 11. $(x^2+xy) \times (-x-y).$ |

12. $(2x^2 + 5ab) \times (2x^2 - 5ab)$. 13. $(3m^2 - 2mn) \times (n^2 - 3mn)$.
 14. $(-\frac{1}{2} + 5x) \times (\frac{1}{4}x - 3)$. 15. $(-2a^2 - \frac{1}{4}) \times (5a - \frac{1}{3})$.

Example 3. Multiply $x^2 + xy - y^2$ by $x^2 - xy + y^2$.

$$\begin{array}{r}
 x^2 + xy - y^2 \\
 x^2 - xy + y^2 \\
 \hline
 x^4 + x^3y - x^2y^2 \\
 -x^3y - x^2y^2 + xy^3 \\
 +x^2y^2 + xy^3 - y^4 \\
 \hline
 x^4 - x^2y^2 + 2xy^3 - y^4
 \end{array}$$

Before the actual process of multiplication, it is generally useful to arrange the multiplicand and the multiplier according to descending or ascending powers of some common letter.

Example 4. Multiply $1 + a^2 + a$ by $-a + 1 + a^2$.

- (i) Arranging according to descending powers of a , (ii) Arranging according to ascending powers of a ,

$$\begin{array}{r}
 a^2 + a + 1 \\
 a^2 - a + 1 \\
 \hline
 a^4 + a^3 + a^2 \\
 -a^3 - a^2 - a \\
 +a^2 + a + 1 \\
 \hline
 a^4 + a^2 + 1
 \end{array}$$

$$\begin{array}{r}
 1 + a + a^2 \\
 1 - a + a^2 \\
 \hline
 1 + a + a^2 \\
 -a - a^2 - a^3 \\
 +a^2 + a^3 + a^4 \\
 \hline
 1 + a^2 + a^4
 \end{array}$$

Example 5. Multiply $3a^2b^2 - ab^3 + 2a^4 - a^3b$ by $-2ab + b^2 - 3a^2$.

Arranging according to descending powers of a ,

$$\begin{array}{r}
 2a^4 - a^3b + 3a^2b^2 - ab^3 \\
 -3a^2 - 2ab + b^2 \\
 \hline
 -6a^6 + 3a^5b - 9a^4b^2 + 3a^3b^3 \\
 -4a^5b + 2a^4b^2 - 6a^3b^3 + 2a^2b^4 \\
 +2a^4b^2 - a^3b^3 + 3a^2b^4 - ab^5 \\
 \hline
 -6a^6 - a^5b - 5a^4b^2 - 4a^3b^3 + 5a^2b^4 - ab^5
 \end{array}$$

Multiply :

16. $1 + a + a^2$ by $1 - a$. 17. $1 - a + a^2$ by $1 + a$.
 18. $a^2 + ab + b^2$ by $a + b$. 19. $a^2 - ab + b^2$ by $a - b$.
 20. $a^4 + a^2b^2 + b^4$ by $a^2 - b^2$. 21. $a^4 - a^2b^2 + b^4$ by $a^2 + b^2$.

22. $7a^2 - 12ab + 9b^2$ by $3a^2 - 4b^2$.
23. $\frac{3}{2}x^2 + xy - \frac{2}{3}y^2$ by $\frac{1}{2}x - \frac{1}{3}y$.
24. $6x^2 - 3xy + 4y^2$ by $2x^2 + 5xy - 3y^2$.
25. $7x^3 - 2x^2y + xy^2 - 4y^3$ by $3x^2 - 4xy + y^2$.
26. $x^2 + xy + y^2$ by $x^2 - xy + y^2$.
27. $x^3 + y^3 + z^3 - 3xyz$ by $x + y + z$.
28. $x^3 + y^3 - z^3 + 3xyz$ by $x + y - z$.
29. $x^4 - x^3y + x^2y^2 - xy^3 + y^4$ by $2x - 3y$.
30. $x^2 - 2xy + y^2 + z^2$ by $x^2 - 2xy + y^2 - z^2$.

Find the product of :

31. $a^3 - 3a^2 + 3a - 1$ and $1 - 2a + a^2$.

Check the result by putting $a = 2$.

32. $3a + 2a^2 + 1$ and $1 + 2a^2 - 3a$.

Check the result by putting $a = -3$.

33. $1 + a + a^3 + a^2 + a^4$ and $1 - a$.
34. $a^2 + b^2 + c^2 - ab - ac - bc$ and $b + a + c$.
35. $y^2 + z^2 + xy + x^2 - yz + xz$ and $x - z - y$.
36. $1 + x^2 + y^2 - xy + x + y$ and $y + x - 1$.

Check the result by putting $x = 2$ and $y = -1$.

37. $xy + yz - zx - y^2$ and $x^2 + xy - yz - z^2$.
38. $1 + 4x^2 + 6x + 2x^3 + x^4$ and $1 + 4x^2 - 6x - 2x^3 + x^4$.
39. $\frac{1}{2}ab + \frac{1}{3}b^2 - \frac{1}{4}a^2$ and $\frac{1}{2}ab + \frac{1}{3}a^2 - \frac{1}{4}b^2$.
40. $6a^2x^2 - 4ax^3 - 4a^3x + x^4 + a^4$ and $a^2 + x^2 - 2ax$.
41. $1 + 3x^6 - 2x^5 - x + x^3$ and $3x - x^3 - 1 + 4x^2$.
42. $3a^2 + a^6 + 1 - 5a - a^3$ and $2a^5 - 1 + a^2 - a^3$.

Example 6. Find the continued product of $a - b$, $a + b$, $a^2 + b^2$ and $a^4 + b^4$.

Here we multiply $a - b$ by $a + b$, the product by $a^2 + b^2$, and that product by $a^4 + b^4$.

$$\begin{array}{r} \text{(i)} \quad a - b \\ \quad \quad a + b \\ \hline a^2 - ab \\ \quad + ab - b^2 \\ \hline a^2 \quad \quad - b^2 \end{array}$$

$$\begin{array}{r} \text{(ii)} \quad a^2 - b^2 \\ \quad \quad a^2 + b^2 \\ \hline a^4 - a^2b^2 \\ \quad + a^2b^2 - b^4 \\ \hline a^4 \quad \quad - b^4 \end{array}$$

$$\begin{array}{r} \text{(iii)} \quad a^4 - b^4 \\ \quad \quad a^4 + b^4 \\ \hline a^8 - a^4b^4 \\ \quad + a^4b^4 - b^8 \\ \hline a^8 \quad \quad - b^8 \end{array}$$

Find the continued product of :

43. $1+a, 1-a, 1+a^2$. 44. $a+1, a+2, a+3$.
 45. $a-2, a+3, a+5$. 46. $x+a, x+b, x+c$.
 47. $(x+y), (x+2y), (x+3y)$. 48. $(a+b), (a+b), (a+b)$.
 49. $(a-b), (a-b), (a-b)$.
 50. $(x+y), (x-y), (x^4+x^2y^2+y^4)$.
 51. $(a^2+ab+b^2), (a^2-ab+b^2), (a^4-a^2b^2+b^4)$.
 52. $(a-b), (a^2+ab+b^2), (a^3+b^3), (a^6+b^6)$.
 53. $(a+b+c), (a-b+c), (a+b-c), (-a+b+c)$.

Find the square of :

54. $a+b+c$. 55. $a-b+c$. 56. $2a-3b+4c$.

Find the cube of :

57. $a+b+c$. 58. $a-b+c$.

Example 7. Find by inspection, the co-efficient of x^2 in the product of $(3x^2-5x+1), (2x-5)$.

Here we have to collect the terms containing x^2 from the partial products.

$2x \times 3x^2$ gives us a term of the 3rd degree, hence neglected.

$2x \times (-5x) = -10x^2$.

$2x \times 1$ gives us a term of the first degree, hence neglected.

$-5 \times 3x^2 = -15x^2$.

$-5 \times (-5x)$ gives us a term of the first degree, hence neglected.

-5×1 is a numerical term, hence neglected.

Thus the term containing x^2 is $\{ (-10x^2) + (-15x^2) \}$

or $-25x^2$ and its co-efficient is -25 .

NOTE.—Generally the above process is done mentally and only the terms containing the required power of x are taken down.

Example 8. If $A=x+y$ and $B=x-y$ find the value of

$$(3A+B)(2A-B).$$

$$3A+B=3(x+y)+(x-y)$$

$$=3x+3y+x-y=4x+2y.$$

$$2A - B = 2(x + y) - (x - y) \\ = 2x + 2y - x + y = x + 3y.$$

$$\therefore (3A + B)(2A - B) = (4x + 2y)(x + 3y) \\ = 4x^2 + 2xy + 12xy + 6y^2 \\ = 4x^2 + 14xy + 6y^2.$$

Find by inspection the co-efficient of :

59. x^2 in the product of $(x^2 - 5x + 2)$, $(x + 4)$.
 60. x^2 „ „ $(3x^2 + 2x - 5)$, $(7x - 4)$.
 61. x „ „ $(2x^2 + x - 1)$, $(x - 3)$.
 62. x „ „ $(6x^2 + xy - y^2)$, $(2x - y)$.
 63. a^2 „ „ $(4a^2 - 7a + 3)$, $(3a + 5)$.
 64. a „ „ $(a - 4)$, $(2a^2 - 9a + 3)$.
 65. x^2 „ „ $(1 - 3x)$, $(4 - 5x^2 + 6x)$.

Find the term containing :

66. x^2 in the continued product of $(x - 1)$, $(x + 2)$, $(x - 3)$.
 67. x „ „ „ $(2x + 5)$, $(3x - 4)$,
 $(2x - 1)$.
 68. x „ „ „ $(4x - 5)$, $(x + 1)$, $(x - 2)$.
 69. y „ „ „ $(1 - 3y + 3y^2)$, $(2 + 3y)$.
 70. If $A = x + y$ and $B = x - y$, find the value of
 $3A^2 + 4AB + 5B^2$.
 71. Find the value of $(A + B)(A - B)$
 when $A = (2x - y)$ and $B = (x - 2y)$.
 *72. Evaluate $P^2 + 2PQ + Q^2$
 when $P = (x + a)^2$, $Q = (x - a)^2$.
 73. Simplify $(2x + 3)(3 - 4x) - (4 + 3x)(2x - 3)$.
 74. Simplify $\frac{3x + 6}{4} \cdot \frac{2x + 15}{3} - \frac{5x - 12}{6} \cdot \frac{x - 2}{2}$.
 *75. Find the value of
 $(2x + 1)^3 - 3(x + 2)(x + 1)^2 + 4(x + 1)(x - 2)^2$.
 76. Simplify $(1 + x)(1 + x + x^2) - (1 - x)(1 - x + x^2)$.
 77. Simplify
 $2(3x^2 + 4x - 1)(7x^2 - 5) - 3(4x^2 + x + 1)(x^2 - 3x - 1)$.
 *78. Multiply $x^{m+n} + 2y^{m-n}$ by $x^{m+n} - y^{m-n}$.

*79. Multiply $2x^2 + 3x + 1$ by $3x^2 + x + 2$, and hence find the product of 231 and 312.

*80. Find the co-efficient of x^3 in the product of $ax^3 + bx^2 + cx + d$ and $px^2 - qx + r$.

81. If $x = a^2 - bc$, $y = b^2 - ca$, $z = c^2 - ab$, find the values of
(i) $ax + by + cz$, (ii) $bx + cy + az$, (iii) $cx + ay + bz$.

DIVISION

10. Just as in Arithmetic $5 \times 3 = 15$ and $15 \div 5 = 3$, similarly, in Algebra if $a \times b = c$, then $c \div a = b$. When division is exact, as $15 \div 5 = 3$ or $c \div a = b$, then obviously

$$\text{Dividend} = \text{Divisor} \times \text{Quotient}.$$

Def. When the product of two quantities is unity, each is called the **reciprocal** of the other; for example, as $a \times \frac{1}{a} = 1$, a is the reciprocal of $\frac{1}{a}$ and $\frac{1}{a}$ is the reciprocal of a .

From the definition, we have

$$1 = \frac{1}{n} \times n \quad \dots \quad \dots \quad \dots \quad (i)$$

Multiplying both sides of (i) by m , we have

$$m \times 1 = m \times \frac{1}{n} \times n,$$

$$\text{or} \quad m = m \times \frac{1}{n} \times n. \quad \dots \quad \dots \quad \dots \quad (ii)$$

Dividing both sides of (ii) by n , we have

$$m \div n = m \times \frac{1}{n} \times n \div n.$$

$$\text{But} \quad n \div n = 1$$

$$\therefore m \div n = m \times \frac{1}{n} \times 1$$

$$= m \times \frac{1}{n}.$$

Or, to divide by a quantity is the same thing as to multiply by its reciprocal.

Thus division is the *inverse* of multiplication.

Cor. $1 \div a = 1 \times \frac{1}{a} = \frac{1}{a}$, and $\frac{1}{a}$ means $1 \div a$.

Just as in Arithmetic $105 \times 7 \div 5 = 105 \div 5 \times 7$

and $105 \div 7 \div 5 = 105 \div 5 \div 7$,

so, in Algebra $a \times b \div c = a \div c \times b$

and $a \div b \div c = a \div c \div b$.

Law. The order of operations in a chain of multiplication and division is immaterial.

Just as in Arithmetic $105 \times (5 \div 7) = 105 \times 5 \div 7$

and $105 \div (5 \div 7) = 105 \div 5 \times 7$,

so, in Algebra $a \times (b \div c) = a \times b \div c$

and $a \div (b \div c) = a \div b \times c$.

Law. When a bracket contains the operations of multiplication and division only, it may be removed, every sign remaining unchanged if \times precedes the bracket and every sign reversed if \div precedes the bracket.

Just as in Arithmetic $(15 + 12 + 9) \div 3 = (15 \div 3) + (12 \div 3) + (9 \div 3)$

so, in Algebra $(a + b + c) \div d = (a \div d) + (b \div d) + (c \div d)$.

Thus, when a multinomial expression is to be divided by a monomial :

divide each term of the dividend by the divisor and take the sum of the partial quotients.

$$\begin{array}{l} \text{Since} \quad +ab = (+a) \times (+b) \\ \therefore \quad \frac{+ab}{+a} = \frac{(+a) \times (+b)}{+a} = +b \end{array} \quad \cdot \quad \cdot \quad (i)$$

$$\begin{array}{l} \text{Since} \quad -ab = (-a) \times (+b) \\ \therefore \quad \frac{-ab}{-a} = \frac{(-a) \times (+b)}{-a} = +b \end{array} \quad \cdot \quad \cdot \quad (ii)$$

In (i) and (ii) the dividend and the divisor have *like signs* and the quotient is *positive*.

Again,

$$+ab = (-a) \times (-b)$$

$$\therefore \frac{+ab}{-a} = \frac{(-a) \times (-b)}{-a} = -b \quad \text{. . . (iii)}$$

Also,

$$-ab = (+a) \times (-b)$$

$$\therefore \frac{-ab}{+a} = \frac{(+a) \times (-b)}{+a} = -b \quad \text{. . . (iv)}$$

In (iii) and (iv) the dividend and the divisor have *unlike signs* and the quotient is *negative*.

Law. Like signs produce + ; unlike signs produce - .

Example 1. Divide $-15a^6b^4$ by $5a^3b^2$.

$$\begin{aligned} (-15a^6b^4) \div (5a^3b^2) \\ &= (-15 \div 5) \times (a^6 \div a^3) \times (b^4 \div b^2) \\ &= -3a^{6-3}b^{4-2} = -3a^3b^2. \end{aligned}$$

After some practice this process should be performed *mentally*.

Example 2. Divide $x^6 - 5ax^5 + 3a^2x^4$ by $-2x^4$.

$$\begin{aligned} \text{The quotient} &= \frac{x^6 - 5ax^5 + 3a^2x^4}{-2x^4} \\ &= \frac{x^6}{-2x^4} - \frac{5ax^5}{-2x^4} + \frac{3a^2x^4}{-2x^4} \\ &= -\frac{1}{2}x^2 + \frac{5}{2}ax - \frac{3}{2}a^2. \end{aligned}$$

EXERCISE 15.

Divide (*mentally*) :

1. $9x^5$ by $3x^3$.
2. $18a^2b^4$ by $-6ab^2$.
3. $-12a^4b^5$ by $-4a^3b^3$.
4. $-15a^3b^2$ by $10a^2b^2$.
5. $-18a^3x^3b^3y^4$ by $-6a^3x^3b^2y^3$.
6. $-35a^3b^3c^4x^4y^3z^3$ by $-5a^3b^3x^3z^3$.
7. $-m^4n^3p^4q^3$ by $m^2n^2p^2q^2$.
8. $-14a^3b^5c^4$ by $+2a^2b^2c^2$.
9. $a^3m^2np^5$ by $-6a^2mp^3$.
10. $-\frac{3}{5}p^3q^4$ by $-\frac{9}{20}p^2q^3$.
11. $\frac{4}{7}a^3b^4c^5m^4$ by $-\frac{8}{21}a^3m^3b^2c^2$.
12. $-\frac{5}{9}a^3b^2c^2x^4$ by $\frac{2}{3}a^2b^3x^2$.

Divide:

13. $-abcd + axcm$ by $-ac$.
14. $6abxy - 4acxz + 2abcxyz$ by $12ax$.
15. $-a^2b^2c - ab^2c^2 - a^2bc^2$ by $-abc$.
16. $25a^4x^5 - 20a^3x^4 - 35a^2x^3$ by $5a^2x^2$.
17. $18a^4b^5x^3 - 12a^2b^3x^4 + 6a^3b^4x^5$ by $-6a^2b^3x^3$.
18. $12a^4b^2c - 18a^2b^4c - 36ab^2c^4$ by $-6ab^2c$.
19. $-\frac{2}{3}abc^3 + \frac{3}{4}ab^3c - \frac{2}{5}a^3bc$ by $-\frac{1}{60}abc$.
20. $-\frac{1}{2}a^2b^3c^4 + \frac{2}{3}b^2c^3a^4 - \frac{3}{4}c^2a^3b^4$ by $-\frac{1}{12}a^2b^2c^2$.

Division of one compound expression by another is analogous to 'Long Division' in Arithmetic, as illustrated below in the division of 483 by 23:

Compact process.

$$\begin{array}{r} 23 \overline{) 483} \begin{array}{l} 21 \\ 46 \\ \hline 23 \\ 23 \end{array} \end{array}$$

Expanded process.

$$\begin{array}{r} 2.10 + 3 \overline{) 4.10^2 + 8.10 + 3} \begin{array}{l} 2.10 + 1 \\ 4.10^2 + 6.10 \\ \hline 2.10 + 3 \\ 2.10 + 3 \end{array} \end{array}$$

In dividing $4x^2 + 8x + 3$ by $2x + 3$ we can proceed in exactly the same way.

$$\begin{array}{r} \text{Thus } 2x + 3 \overline{) 4x^2 + 8x + 3} \begin{array}{l} 2x + 1 \\ 4x^2 + 6x \\ \hline 2x + 3 \\ 2x + 3 \end{array} \end{array}$$

Explanation. The first term of the dividend is divided by the first term of the divisor. Thus $4x^2 \div 2x = 2x$. This gives us the first term of the quotient. The whole divisor is multiplied by $2x$ and the product is subtracted from the dividend. Thus we have the remainder $2x + 3$. We treat it as the new dividend and divide its first term by the first term of the divisor. Thus $2x \div 2x = 1$. This gives us the second term of the quotient. The whole divisor is multiplied by 1 and the product is subtracted from $2x + 3$. There being no remainder, the process is complete, and the quotient is $2x + 1$.

Example 1. Divide $4x^3 + 7x^2 - 3x - 15$ by $4x - 5$.

$$\begin{array}{r}
 4x-5 \overline{) 4x^3 + 7x^2 - 3x - 15} \left(x^2 + 3x + 3 \right. \\
 \underline{4x^3 - 5x^2} \\
 12x^2 - 3x \\
 \underline{12x^2 - 15x} \\
 12x - 15 \\
 \underline{12x - 15} \\
 0
 \end{array}$$

The required quotient is $x^2 + 3x + 3$.

In this example the dividend and the divisor are already arranged in descending powers of x ; however, if they are not arranged in descending or ascending powers of a common letter, we have to arrange them so, before doing the actual process.

Example 2. Divide $2 - 3x^2 + x - x^3 + x^4$ by $x^2 + 2 - 3x$.

Arrange the expressions according to descending powers of x .

$$\begin{array}{r}
 x^2 - 3x + 2 \overline{) x^4 - x^3 - 3x^2 + x + 2} \left(x^2 + 2x + 1 \right. \\
 \underline{x^4 - 3x^3 + 2x^2} \\
 2x^3 - 5x^2 + x \\
 \underline{2x^3 - 6x^2 + 4x} \\
 x^2 - 3x + 2 \\
 \underline{x^2 - 3x + 2} \\
 0
 \end{array}$$

The required quotient $= x^2 + 2x + 1$.

Thus the process of Algebraic long division can be stated as follows :

Rule. (i) Arrange the dividend and the divisor in descending or ascending powers of a common letter.

(ii) Divide the term on the left of the dividend by the term on the left of the divisor. The result is the first term of the quotient.

(iii) Multiply the whole divisor by this quotient and subtract the product from the dividend.

(iv) Bring down to the remainder as many terms of the dividend as may be necessary to form a new dividend, and go on repeating the above steps till all the terms of the dividend have been used.

Divide :

21. $x^2 + 8x + 15$ by $x + 5$. 22. $x^2 + 2x - 15$ by $x - 3$.
 23. $x^2 - 8x + 15$ by $x - 5$. 24. $x^2 - 14x + 49$ by $x - 7$.
 25. $2x^2 - 3x - 2$ by $2x + 1$. 26. $10x^2 - 14x - 12$ by $2x - 4$.
 27. $3m^2 + m - 2$ by $m + 1$. 28. $3a^2 + 10a + 3$ by $3a + 1$.
 29. $6a^2 - 7a - 3$ by $3a + 1$. 30. $4p^2 - 4p - 3$ by $2p - 3$.
 31. $5m^2 - 17m + 6$ by $5m - 2$. 32. $-21a^2 + a + 10$ by $-7a + 5$.
 33. $1 - x + x^2 - x^3$ by $1 + x^2$. 34. $3a - 3a^2 + a^3 - 1$ by $a - 1$.
 35. $3x^3 - 2y^3 + 9xy^2 - 11x^2y$ by $3x - 2y$.
 36. $3a^3 - a^2b - 2b^3 - 5ab^2$ by $2b + 3a$.
 37. $6a^4 - a^3 + 5a - 6 + 4a^2$ by $3a^2 - 2 + a$.
 38. $a^4 - 4a^3 - 18a^2 - 11a + 2$ by $a^2 - 7a + 1$.
 39. $12a^4 + a^3 - 8a^2 + 7a - 2$ by $3a^2 - 2a + 1$.
 40. $2a^4 + a^2 - 3a^3 - 9 + 9a$ by $2a^2 - 3 + a$.
 41. $4x - 1 - 8x^2 - 6x^4 + 11x^3$ by $1 - x + 3x^2$.
 42. $3 - a^2 - 14a - 4a^3 + a^4$ by $a + 3 + a^2$.
 43. $1 + a + a^2 + a^3 + a^4 + a^5$ by $1 + a^2 + a^4$.
 44. $p^4 - 3p^3q + pq^3 + 3p^2q^2 - 6q^4$ by $p^2 - pq - 2q$.
 45. $3x^5 + 3x^4 + 2x^3 + 1$ by $3x^3 - x + 1$.

Example 3. Divide $1 - a^4$ by $1 - a$.

$$\begin{array}{r}
 1-a \overline{) 1-a^4} \left(1+a+a^2+a^3 \right. \\
 \underline{1-a} \\
 a-a^4 \\
 a-a^2 \\
 \underline{a^2-a^4} \\
 a^2-a^3 \\
 \underline{a^3-a^4} \\
 a^3-a^4
 \end{array}$$

The required quotient $= 1 + a + a^2 + a^3$.Example 4. Divide $a^4 + a^2b^2 + b^4$ by $a^2 - ab + b^2$.

$$\begin{array}{r}
 a^2-ab+b^2 \overline{) a^4+a^2b^2+b^4} \left(a^2+ab+b^2 \right. \\
 \underline{a^4-a^3b+a^2b^2} \\
 a^3b+b^4 \\
 a^3b-a^2b^2+ab^3 \\
 \underline{a^2b^2-ab^3+b^4} \\
 a^2b^2-ab^3+b^4
 \end{array}$$

The required quotient $= a^2 + ab + b^2$.

Divide :

46. $x^3 + y^3$ by $x + y$.
 47. $x^5 + y^5$ by $x + y$.
 48. $x^5 - y^5$ by $x - y$.
 49. $x^8 - y^8$ by $x + y$.
 50. $x^9 - 1$ by $x^2 + x + 1$.
 51. $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$.
 52. $x^6 - 1$ by $x^3 + 2x^2 + 2x + 1$.
 53. $x^6 - 2x^3 + 1$ by $x^2 - 2x + 1$.
 54. $x^9 + y^9$ by $x^3 + y^3$.
 55. $x^3 + 3xy + y^3 - 1$ by $x + y - 1$.

Check the result by putting $x = 2, y = 1$.

56. $x^6 - y^6$ by $x^3 + 2x^2y + 2xy^2 + y^3$.
 57. $x^{12} - 2x^6y^6 - 3y^{12}$ by $x^4 - x^2y^2 + y^4$.
 58. $a^4 - \frac{1}{81}b^4$ by $a - \frac{1}{3}b$.

Check the result by putting $a = -1$ and $b = -2$.

59. $x^4 - \frac{5}{4}x^3 + \frac{11}{8}x^2 - \frac{1}{2}x$ by $x^2 - \frac{1}{2}x$.
 60. $\frac{1}{4}x^2 - \frac{1}{9}y^2 - \frac{1}{36}z^2 + \frac{1}{9}yz$ by $\frac{1}{2}x - \frac{1}{3}y + \frac{1}{6}z$.
 61. $x^4 + y^4 - z^4 + 2x^2y^2 - 2z^2 - 1$ by $x^2 + y^2 - z^2 - 1$.
 62. $a^3 - b^3 - c^3 - 3abc$ by $a^2 + b^2 + c^2 + ab + ac - bc$.
 63. $x^3 + 8y^3 - 27z^3 + 18xyz$ by $x - 3z + 2y$.
 64. $a^3 + 8b^3 + 27c^3 - 18abc$ by $a^2 + 4b^2 + 9c^2 - 2ab - 6bc - 3ca$.
 65. $x^8 + x^6y^2 + x^4y^4 + x^2y^6 + y^8$ by $x^4 - x^3y + x^2y^2 - xy^3 + y^4$.
 66. $x^{3n} - y^{3n}$ by $x^n - y^n$.

INEXACT DIVISION

Example 5. Divide $x^2 + 7x + 12$ by $x + 2$.

$$\begin{array}{r} x+2 \overline{) x^2 + 7x + 12} \quad (x+5 \\ \underline{x^2 + 2x} \\ 5x + 12 \\ \underline{5x + 10} \\ 2 \end{array}$$

It is an instance of inexact division, where 2 is left as the remainder and $x^2 + 7x + 12 = (x + 2)(x + 5) + 2$. In all similar cases of inexact division, if we denote the dividend by D , the divisor by d , the quotient by Q and the remainder by R , we have

$$D = dQ + R,$$

and the complete quotient $= Q + \frac{R}{d}$.

Example 6. Divide $x^4 - 3x^3 - 13x^2 + 12x + 4$ by $x^2 - x + 2$.

First method. Arrange the divisor and the dividend according to *descending powers of x*.

$$\begin{array}{r}
 x^2 - x + 2 \overline{) x^4 - 3x^3 - 13x^2 + 12x + 4} \quad (x^2 - 2x - 17 \\
 \underline{x^4 - x^3 + 2x^2} \\
 -2x^3 - 15x^2 + 12x \\
 \underline{-2x^3 + 2x^2 - 4x} \\
 -17x^2 + 16x + 4 \\
 \underline{-17x^2 + 17x - 34} \\
 -x + 38
 \end{array}$$

The quotient $= x^2 - 2x - 17$ and the remainder $= -x + 38$.

The complete quotient $= x^2 - 2x - 17 + \frac{-x + 38}{x^2 - x + 2}$.

Second method. Arrange the divisor and the dividend according to *ascending powers of x*.

$$\begin{array}{r}
 2 - x + x^2 \overline{) 4 + 12x - 13x^2 - 3x^3 + x^4} \quad (2 + 7x - 4x^2 - 7x^3 \\
 \underline{4 - 2x + 2x^2} \\
 14x - 15x^2 - 3x^3 \\
 \underline{14x - 7x^2 + 7x^3} \\
 -8x^2 - 10x^3 + x^4 \\
 \underline{-8x^2 + 4x^3 - 4x^4} \\
 -14x^3 + 5x^4 \\
 \underline{-14x^3 + 7x^4 - 7x^5} \\
 -2x^4 + 7x^5
 \end{array}$$

The quotient $= 2 + 7x - 4x^2 - 7x^3$ and the remainder $= -2x^4 + 7x^5$.

The complete quotient $= 2 + 7x - 4x^2 - 7x^3 + \frac{-2x^4 + 7x^5}{2 - x + x^2}$.

NOTE. It may be observed that the results in *inexact division* by the above two methods are different.

Divide and give the complete quotient in each case :

67. $a^2 + b^2 + 2ab + 3$ by $1 + a + b$.

68. $a^3 + 7a^2 + 21a + 17$ by $a^2 + 5a + 6$.

69. $a^4 + 7a^3 + 14a^2 + 9a + 4$ by $a^2 + 3a + 1$.

70. $2a^4 + 3ab^3 - 4a^2b^2 + a^3b - 5b^4$ by $a^2 - 2b^2 + ab$.

Divide up to 4 terms; give the remainder and the complete quotient in each case :

71. 1 by $1+x$. 72. $1-2a$ by $1+3a$. 73. $1-a$ by $1-2a+2a^2$.

Prove that :

$$74. \frac{a^2 + 11a + 35}{a + 5} = a + 6 + \frac{5}{a + 5}.$$

$$75. \frac{x^2 - 9x + 25}{x - 4} = x - 5 + \frac{5}{x - 4}.$$

$$76. \frac{a^2 - 13a + 45}{a - 7} = a - 6 + \frac{3}{a - 7}.$$

$$77. \frac{x^2 - 16x + 60}{x - 9} = x - 7 - \frac{3}{x - 9}.$$

78. $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$ is the product of two expressions of which one is $x^2 - 2x + 1$. Find the other.

79. Find the divisor when the dividend is $x^3 - 6x^2 + 12x - 5$, the quotient $x - 3$ and the remainder $x + 1$.

80. What must be subtracted from $12a^2 + 8a - 9$, so that it may be exactly divisible by $2a - 1$?

81. What must be added to $3a^3 - 8a^2 + 7a$, so that it may be exactly divisible by $3a - 2$?

82. What must be added to $3a^3 + 7a^2 - 6a - 5$, so that it may be exactly divisible by $a^2 + 2a - 1$?

*83. Find the condition that $x^4 - 3x^2 + m$ may be exactly divisible by $x^2 + 1$?

*84. For what value of k is $9a^3 - 6a^2 + 3a - k$ exactly divisible by $a^2 - 2a + 3$?

*85. For what value of x will $2x^3 + 5x^2 - mx + 4$ be exactly divisible by $x^2 + 2x - 1$?

*86. Find a and b in order that $x^4 - 2x^3 + 8x^2 + ax + b$ may be exactly divisible by $x^2 + x + 5$.

CHAPTER III

SIMPLE EQUATIONS AND PROBLEMS

1. If we *simplify* $(3x+4)+2(x-1)$, the result is $5x+2$,
or $(3x+4)+2(x-1)=5x+2$.

Here, if we put $x=1$,

$$\begin{aligned}\text{the left-hand side} &= (3.1+4)+2(1-1)=7 \\ \text{and the right-hand side} &= 5.1+2=7.\end{aligned}$$

Thus the left-hand side = the right-hand side.

Again, if we put $x=2$,

$$\begin{aligned}\text{the left-hand side} &= (3.2+4)+2(2-1)=12 \\ \text{and the right-hand side} &= 5.2+2=12.\end{aligned}$$

Thus the left-hand side = the right-hand side.

If in the above we substitute 3, 4, 5, 6,...*any number* for x , the left-hand side is always = the right-hand side.

Def. When one expression is equal to another expression for all values of the letter or letters involved (or all values of the quantities used), it is said to form an **Identical Equation** or simply an **Identity**.

Now if we substitute in $(3x+4)+2(x-1)=4x+7$, different values of x , say 1, 2, 3, 4, 5, 6,...etc., we find that the left-hand side = the right-hand side *only when* $x=5$ and for no other values of x . In this case the equality is obviously *not universal* but *conditional* and holds good only when $x=5$. Such an equality is called **Conditional Equation** or simply an **Equation**.

Def. When one expression is equal to another expression for particular value or values of the quantity or quantities used, an **Equation** is formed.

The sign $=$ is used for an equation and \equiv for an identity.

It is useful to note the effect of ordinary processes of addition, subtraction, multiplication or division, as illustrated below in the four examples:

(i) When we add $x^2 + xy + y^2$ and $x^2 - xy + y^2$, the result is $2x^2 + 2y^2$ for all values of x and y ,

$$\text{or } (x^2 + xy + y^2) + (x^2 - xy + y^2) \equiv 2x^2 + 2y^2.$$

(ii) When we subtract $x^2 - xy + y^2$ from $x^2 + xy + y^2$, the result is $2xy$ for all values of x and y ,

$$\text{or } (x^2 + xy + y^2) - (x^2 - xy + y^2) \equiv 2xy.$$

(iii) When we multiply $x^2 + xy + y^2$ and $x^2 - xy + y^2$, the result is $x^4 + x^2y^2 + y^4$ for all values of x and y

$$\text{or } (x^2 + xy + y^2)(x^2 - xy + y^2) \equiv x^4 + x^2y^2 + y^4,$$

(iv) When we divide $x^4 + x^2y^2 + y^4$ by $x^2 + xy + y^2$ the result is $x^2 - xy + y^2$ for all values of x and y ,

$$\text{or } (x^4 + x^2y^2 + y^4) \div (x^2 + xy + y^2) \equiv x^2 - xy + y^2.$$

Example 1. Prove the identity

$$(x+y)^2 - 4xy \equiv (x-y)^2 \text{ and verify it when}$$

(i) $x=3$ and $y=1$, (ii) $x=2$ and $y=1$, (iii) $x=4$ and $y=2$.

The left-hand side $= x^2 + 2xy + y^2 - 4xy = x^2 - 2xy + y^2$ and the right-hand side $= x^2 - 2xy + y^2$.

Since by *expanding*, both sides can be transformed into the same expression, \therefore the identity is true.

(i) When $x=3$ and $y=1$,

$$\text{the left-hand side} = (3+1)^2 - 4.3.1 = 16 - 12 = 4$$

$$\text{and the right-hand side} = (3-1)^2 = 2^2 = 4.$$

(ii) When $x=2$ and $y=1$,

$$\text{the left-hand side} = (2+1)^2 - 4.2.1 = 9 - 8 = 1$$

$$\text{and the right-hand side} = (2-1)^2 = 1^2 = 1.$$

(iii) When $x=4$ and $y=2$,

$$\text{the left-hand side} = (4+2)^2 - 4.4.2 = 36 - 32 = 4$$

$$\text{and the right-hand side} = (4-2)^2 = 2^2 = 4.$$

Thus in each case, the left-hand side = the right-hand side.

EXERCISE 16.

Prove the following identities and verify each when

(i) $a=4$ and $b=2$, (ii) $a=3$ and $b=1$, (iii) $a=5$ and $b=3$:

$$1. (a+b)^2 \equiv a^2 + 2ab + b^2. \quad 2. (a-b)^2 \equiv a^2 - 2ab + b^2.$$

$$3. (a+b)(a-b) \equiv a^2 - b^2. \quad 4. (a+b)^2 - (a-b)^2 \equiv 4ab.$$

$$5. (a+b)^2 + (a-b)^2 \equiv 2(a^2 + b^2).$$

$$6. (a+b)(a^2 - ab + b^2) \equiv a^3 + b^3.$$

$$7. (a-b)(a^2 + ab + b^2) \equiv a^3 - b^3.$$

$$8. (a+3b)^2 + (a-3b)^2 \equiv 2(a+3b)(a-3b) + 36b^2.$$

In a conditional equation like $5x + 2 = 4x + 7$, the letter x stands for a number whose value is not known, and this x is generally called the **unknown**. Any particular value of the unknown which makes the two sides of an equation equal is said to **satisfy** the equation and is called the **root** or the **solution** of the equation. To *solve* an equation is to find its root.

When an equation, reduced to its simplest form, contains no power of the unknown higher than the first, it is called a **Simple Equation**.

Find by *inspection* the values of x which satisfy the following equations :

$$9. 6x = 12. \quad 10. 8x = 32. \quad 11. 4x = 1. \quad 12. 5x = 0.$$

$$13. x + 5 = 16. \quad 14. x + 7 = 18. \quad 15. \frac{x}{2} = 4. \quad 16. \frac{x}{3} = 6.$$

$$17. x + 3 = 7. \quad 18. x - 3 = 6. \quad 19. x - 5 = 8. \quad 20. x - 5 = 0.$$

$$21. x - 7 = 0. \quad 22. x + 4 = 0. \quad 23. x + 6 = 0. \quad 24. x + 8 = 0.$$

$$25. \frac{2x}{3} = 6, \text{ find } 2x \text{ and then } x.$$

$$26. \frac{3x}{5} = 12, \text{ find } 3x \text{ and then } x.$$

$$27. \frac{x+1}{2} = 4, \text{ find } x+1 \text{ and then } x.$$

$$28. \frac{x-1}{3} = 6, \text{ find } x-1 \text{ and then } x.$$

$$29. ax = c, \text{ find } x.$$

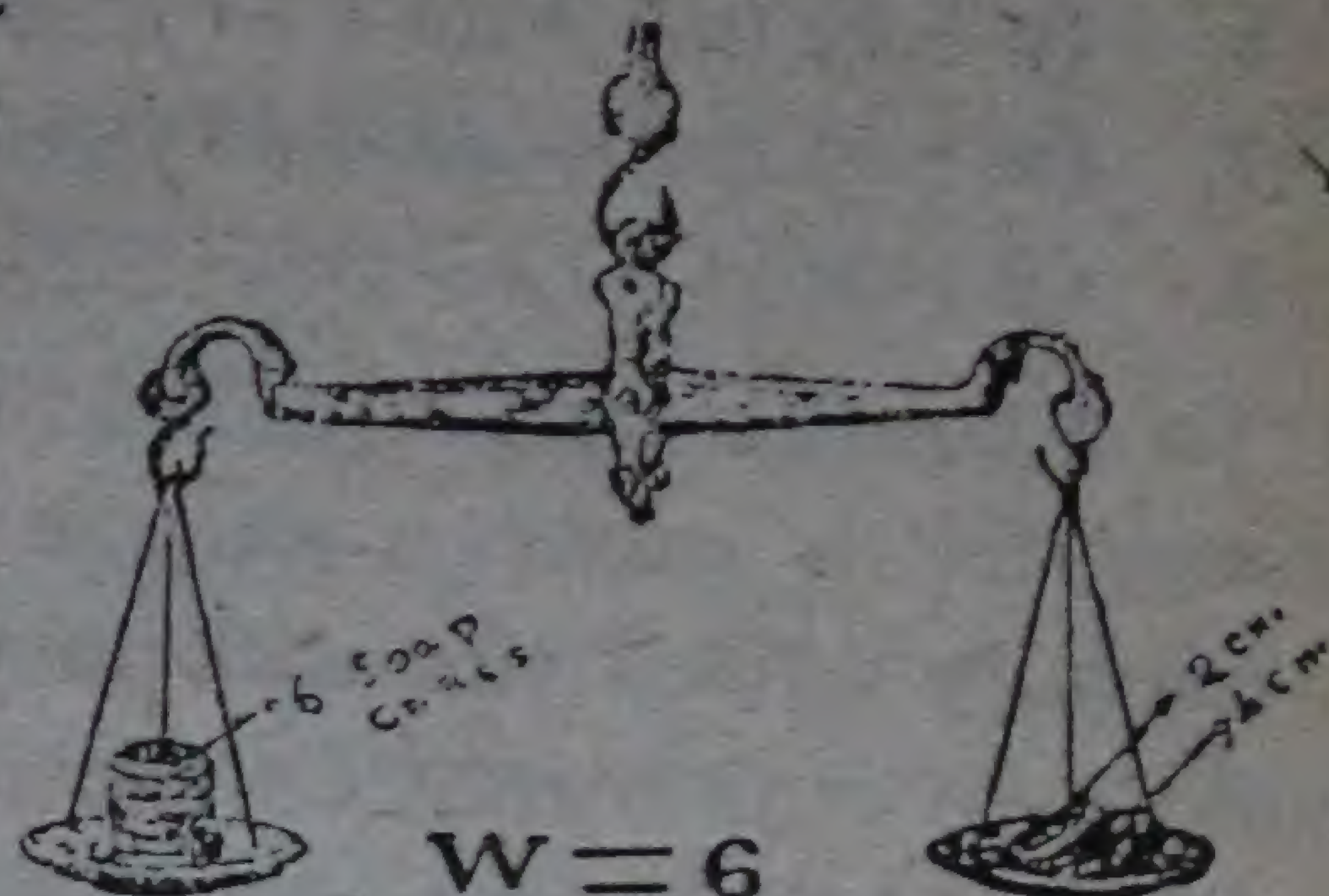
$$30. \frac{x}{a} = b, \text{ find } x.$$

$$31. x + a \equiv b, \text{ find } x.$$

$$32. x - a \equiv b, \text{ find } x.$$

2. Methods of solving simple equations.

An *equation* may be compared to a *balance*. An equation has two sides connected by the sign of equality ; a balance has two scale-pans connected by a beam. The two sides of an equation are equal, and the two scale-pans, with equal weights, keep the beam horizontal.



Suppose there are a few soap cakes in the left-hand scale-pan, and they are balanced by 6 chattak weights placed in the right-hand scale-pan.

Representing it symbolically, we have

$$w = 6.$$

Now, if a certain amount of soap (say 2 ch.) be added to or removed from the left-hand scale-pan, weights equal to that amount must be added to or removed from the right-hand scale-pan to keep the beam horizontal.

Representing it symbolically, we have

$$w + 2 = 6 + 2 \quad . \quad . \quad . \quad . \quad (i)$$

and

$$w - 2 = 6 - 2 \quad . \quad . \quad . \quad . \quad (ii)$$

If the amount of soap be doubled or trebled, the weights will have to be doubled or trebled to keep the beam horizontal.

Representing it symbolically, we have

$$\begin{aligned} & w \times 2 = 6 \times 2 \\ & w \times 3 = 6 \times 3 \end{aligned} \quad \left. \vphantom{\begin{aligned} & w \times 2 = 6 \times 2 \\ & w \times 3 = 6 \times 3 \end{aligned}} \right\} . \quad . \quad . \quad . \quad (iii)$$

Again, if the amount of soap be halved or reduced to one-third, the weights will have to be halved or reduced to one-third to keep the beam horizontal.

Representing it symbolically, we have

$$\text{and} \quad \left. \begin{array}{l} \frac{w}{2} = \frac{6}{2} \\ \frac{w}{3} = \frac{6}{3} \end{array} \right\} \quad \cdot \quad \cdot \quad \cdot \quad (iv)$$

From these illustrations and from the process of oral exercises from 9 to 32 done above it is clear that the following axioms are very useful in the solution of equations:

If equals be added to equals, the sums are equal.

[Axiom I]

If equals be subtracted from equals, the remainders are equal.

[Axiom II]

If equals be multiplied by equals, the products are equal.

[Axiom III]

If equals be divided by equals, the quotients are equal.

[Axiom IV]

Example 1. Solve the equation $x - 7 = 10$.

Adding 7 to both sides, we have

$$x - 7 + 7 = 10 + 7$$

[Axiom I]

$$\therefore x = 17.$$

Example 2. Solve the equation $x + 9 = 13$.

Subtracting 9 from both sides, we have

$$x + 9 - 9 = 13 - 9$$

[Axiom II]

$$\therefore x = 4.$$

Example 3. Solve the equation $\frac{x}{5} = 4$.

Multiplying both sides by 5, we have

$$\frac{x}{5} \times 5 = 4 \times 5$$

[Axiom III]

$$\therefore x = 20.$$

Example 4. Solve the equation $3x = 12$.

Dividing both sides by 3, we have

$$\frac{3x}{3} = \frac{12}{3}$$

[Axiom IV]

$$\therefore x = 4.$$

Example 5. Solve the equation $3y + 2 = 17$.

Subtracting 2 from both sides, we have

$$3y + 2 - 2 = 17 - 2 \quad . \quad . \quad [\text{Axiom II}]$$

$$\therefore 3y = 15.$$

Dividing both sides by 3, we have

$$\frac{3y}{3} = \frac{15}{3} \quad . \quad . \quad [\text{Axiom IV}]$$

$$\therefore y = 5.$$

Example 6. Solve the equation $\frac{z}{4} - 6 = 10$.

Adding 6 to both sides, we have

$$\frac{z}{4} - 6 + 6 = 10 + 6 \quad . \quad . \quad [\text{Axiom I}]$$

$$\therefore \frac{z}{4} = 16.$$

Multiplying both sides by 4, we have

$$\frac{z}{4} \times 4 = 16 \times 4 \quad . \quad . \quad [\text{Axiom III}]$$

$$\therefore z = 64.$$

EXERCISE 17.

Solve the following equations and do *full process* :

1. $3x = 18$.

2. $5x = 35$.

3. $7x = 21$.

4. $11x = 44$.

5. $\frac{y}{4} = 7$.

6. $\frac{y}{6} = 3$.

7. $\frac{y}{8} = 5$.

8. $\frac{y}{12} = 4$.

9. $\frac{1}{3}z = 6$.

10. $\frac{1}{5}z = 2$.

11. $x - 5 = 13$.

12. $x - 3 = 12$.

13. $x - 9 = 4$.

14. $x - 14 = 3$.

15. $x - 7 = 6$.

16. $x - 2 = 9$.

17. $x + 4 = 11$.

18. $x + 6 = 15$.

19. $x + 8 = 17$.

20. $x + 10 = 21$.

21. $2y + 3 = 15$.

22. $3z + 4 = 22$.

23. $5x + 2 = 27$.

24. $4x + 5 = 21$.

- | | |
|---|--|
| 25. $3y - 1 = 8.$ | 26. $7y - 6 = 15.$ |
| 27. $6z - 3 = 33.$ | 28. $5z - 7 = 38.$ |
| 29. $3x - 5 - x = 13.$ | 30. $5x - 9 - 2x = 12.$ |
| 31. $2x + 7 + x = 10.$ | 32. $4x + 3 + x = 15.$ |
| 33. $6x - 2 = 3x + 7.$ | 34. $4z + 6 = 2z + 11.$ |
| 35. $5y - 1 = 2y + 9.$ | 36. $4y + 7 = y + 17.$ |
| 37. $\frac{x}{2} - 1 = 2.$ | 38. $\frac{z}{3} + 2 = 4.$ |
| 39. $\frac{z}{6} + \frac{1}{6} = \frac{5}{6}.$ | 40. $\frac{z}{5} - \frac{1}{5} = \frac{2}{5}.$ |
| 41. $\frac{x}{12} + \frac{5}{4} = 2.$ | 42. $\frac{5x}{6} - \frac{1}{4} = \frac{3}{8}.$ |
| 43. $\frac{3x}{8} + \frac{1}{6} = \frac{3}{4}.$ | 44. $\frac{5x}{12} - \frac{3}{8} = \frac{1}{2}.$ |
| 45. $5x - 8 + 3x = 12 - 4x.$ | 46. $17x - 3 + 2x = 7x + 18 + 5x.$ |
| 47. $2x - (x + 4) = 9.$ | 48. $5y - (3y - 2) = 22.$ |
| 49. $17x + (8x - 11) = 39.$ | 50. $z - (6 - 4z) + 5 = 14.$ |
| 51. $7y + (5y - 9) - 6 = 21.$ | 52. $4x - (x + 5) + 3 = 0.$ |
| 53. $9x + (12 - 3x) - 26 = 0.$ | 54. $2y - (7 - 5y) - 24 = 0.$ |

3. **Transposition.** Consider the equation

$$2x - 7 = 5 \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (i)$$

Adding 7 to both sides, we have

$$\begin{aligned} 2x - 7 + 7 &= 5 + 7 && \cdot \quad \cdot \quad \cdot \quad \cdot \quad [Axiom 1] \\ \therefore 2x &= 5 + 7 && \cdot \quad \cdot \quad \cdot \quad \cdot \quad (ii) \end{aligned}$$

Comparing (i) and (ii), we notice that -7 *disappears* from the left-hand side and *appears* on the right-hand side *with its sign changed*.

Again, let us consider the equation

$$5x = 15 + 2x \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (iii)$$

Subtracting $2x$ from both sides, we have

$$\begin{aligned} 5x - 2x &= 15 + 2x - 2x && [Axiom II] \\ \therefore 5x - 2x &= 15 && \cdot \quad \cdot \quad \cdot \quad \cdot \quad (iv) \end{aligned}$$

Comparing (iii) and (iv), we notice that $+2x$ *disappears* from the right-hand side and *appears* on the left-hand side *with its sign changed*.

By taking a few more similar equations and proceeding as above, we notice that the process is based on the application of addition and subtraction axioms but for all practical purposes it assumes the form of the following rule:—

We may transpose a term from one side of an equation to the other, if at the same time we change the sign of the term from $+$ to $-$ or from $-$ to $+$, as the case may be.

For the sake of convenience this process is called the process of **transposition**.

Abbreviations. R.H.S. stands for the *right-hand side* and L.H.S. stands for the *left-hand side*.

Example 7. Solve $11x - 5 = 4x + 9$.

Transposing $4x$ and 5 , we have

$$11x - 4x = 9 + 5$$

$$\therefore 7x = 14 \quad \text{and} \quad x = 2.$$

[**Verification.** Put 2 for x in the above equation.

$$\text{L.H.S.} = 11 \times 2 - 5 = 17.$$

$$\text{R.H.S.} = 4 \times 2 + 9 = 17.$$

Thus $\text{L.H.S.} = \text{R.H.S.}$, and the solution is correct.]

Example 8. Solve $5x - 18 - 2x = x + 2 - 3x$.

Transposing all the terms containing x to the left-hand side and all other terms to the right-hand side, we have

$$5x - 2x - x + 3x = +2 + 18$$

$$\therefore 5x + 3x - 3x = 20$$

$$\therefore 5x = 20$$

$$\text{or} \quad x = 4.$$

[**Verification.** Put 4 for x in the above equation.

$$\text{L.H.S.} = 5 \times 4 - 18 - 2 \times 4 = -6.$$

$$\text{R.H.S.} = 4 + 2 - 3 \times 4 = -6.$$

Thus $\text{L.H.S.} = \text{R.H.S.}$, and the solution is correct.]

NOTE. In practice it is *usual* to bring all the terms containing x to the left-hand side and all other terms to the right-hand side.

Example 9. Solve $4(2x+5)-2(x+1)=0$.

Removing the brackets, we have

$$8x+20-2x-2=0$$

or $6x+18=0$.

Transposing +18, we have

$$6x=-18$$

$\therefore x=-3$.

[**Verification.** Put -3 for x in the above equation.

$$\begin{aligned}\text{L.H.S.} &= 4 \{ 2 \times (-3) + 5 \} - 2(-3+1) \\ &= 4 \times (-1) - 2 \times (-2) = -4 + 4 = 0.\end{aligned}$$

$$\text{R.H.S.} = 0.$$

Thus L.H.S. = R.H.S., and the solution is correct.]

Example 10. Solve $(x-3)(2x-1)=5+(2x+3)(x-2)$.

Simplifying both sides of the equation, we have

$$2x^2-x-6x+3=5+2x^2-4x+3x-6$$

$$\therefore 2x^2-x-6x-2x^2+4x-3x=5-6-3 \quad \dots \quad [\text{Transposition.}]$$

or $-6x=-4$

$\therefore 6x=4$

or $x=\frac{2}{3}$.

[**Verification.** Put $\frac{2}{3}$ for x in this equation.

$$\text{L.H.S.} = (\frac{2}{3}-3)(2 \times \frac{2}{3}-1) = (-\frac{7}{3})(\frac{1}{3}) = -\frac{7}{9}.$$

$$\begin{aligned}\text{R.H.S.} &= 5 + (2 \times \frac{2}{3} + 3)(\frac{2}{3}-2) \\ &= 5 + \frac{13}{3} \times (-\frac{4}{3}) = 5 - \frac{52}{9} = -\frac{7}{9}.\end{aligned}$$

Thus L.H.S. = R.H.S., and the solution is correct.]

NOTE. It will be found useful to remember the following hints :

(i) We may change the signs of *all* the terms on both sides of an equation without altering the equality. Thus, in $-x+5=-8$, if we divide both sides by -1 , we get $x-5=8$.

(ii) We may interchange the position of the two sides of an equation without changing the signs.

Thus, $11=x-3$ may be written as $x-3=11$.

(iii) We may cancel the same term from both sides of an equation. Thus $2x-5=6+x-5$ is the same thing as $2x=6+x$, and $17x-4+2x=5x+6+2x$ is the same thing as $17x-4=5x+6$.

In solving the following equations apply the *method of transposition*, and *verify* each solution :

55. $3x - 5 = 2x - 3.$

56. $4x - 1 = 6x + 5.$

57. $2x + 3 = x - 8.$

58. $5x + 7 = 2x + 3.$

59. $11 - 2x = 15 - x.$

60. $7 + 3x = 12 - 2x.$

61. $15 + 3x = 8 + 2x.$

62. $9 - 5x = 13 + 3x.$

63. $-4 + 2x = 6 - 3x.$

64. $12 - 4x = -5 + 2x.$

65. $-3y - 5 = -7y + 1.$

66. $-5 + 7x = -11 - 2x.$

67. $3(5x + 6) = 4(2 - x).$

68. $9x + 20 - 4x = 45.$

69. $3 + 3x - 4 = 5x - 1.$

70. $7x - 6 + x = 16 + 3x.$

71. $-5x + 2 = 8x - 20 - 4x.$

72. $5x + 23 + 11x = -8x + 46 + x.$

73. $5x - 9 + 3x - 7 = 0.$

74. $3x - 6 + 5x - 12 = 0.$

75. $9x - 8 + 2x - 4 - 8x + 10 = 0.$

76. $4x + 20 - 8(13 - x) = 0.$

77. $6x - 2(4 - 3x) = 7 - 3(17 - x).$

78. $3(x - 5) - 7(2x + 3) = 0.$

79. $4(2x - 3) + 5(x - 2) = 0.$

80. $2(3x - 1) - 6(2x - 5) = 0.$

81. $-7(12 + 3x) + 5(x - 2) = 0.$

82. $-3(5 - 4x) + 2(3x - 1) = 0.$

83. $6 - \{ 2x - (3x - 4) - 1 \} = 0.$

84. $2x - (3 - 5x) + 2 = 4x - 7.$

85. $(2x - 7)(3x + 1) = (2x + 5)(3x + 1).$

86. $(4x - 3)(2x - 1) = (2x + 5)(4x - 1).$

87. $(4x + 5)(3x - 2) = (6x - 1)(2x - 5).$

88. $(x - 6)(12x + 1) = (3x + 2)(4x - 1).$

89. $(3x + 2)(8x + 5) = (6x - 1)(4x + 7).$

90. $-(x - 8)(x + 12) = (x + 1)(6 - x).$

91. $-(7 - 6x)(3 - 2x) = -(4x - 3)(3x - 2).$

92. $(x + 3)^2 + (4 - x)^2 = 2x^2 + x.$

[Hint. $(x + 3)^2 = (x + 3)(x + 3)$ and $(4 - x)^2 = (4 - x)(4 - x)$].

93. $(x + 5)^2 - (2 - x)^2 = 7.$

94. $(2x+1)^2 - 8 = (2x-1)^2$.
95. $14 - 5(x-4)(x+8) = 188 - 5(x+3)^2$.
96. $9(x-2)^2 + 3(x-4)^2 = (3x-7)(4x-19) + 42$.
97. Find the value of x which will make $5(x-4)+2$ equal to 52.
98. Find the value of x which will make $2(x-3)+7$ equal to zero.
99. For what value of m will $(2m-3)(m+4)+5-2m^2$ be equal to zero?
100. For what values of x are the following statements true:
- (i) $(12x+5)(x-3) = (3x+4)(4x+3) + 29$?
 - (ii) $(2x-1)(6x+5) = 4(3x-2)(x+1) + 3$?
 - (iii) $(2x-1)(3x-4) = (6x-5)(x-1)$?

EQUATIONS INVOLVING NUMERICAL DENOMINATORS

4. To get rid of the numerical denominators in an equation, multiply both sides of it by the L. C. M. of the denominators.

Example 1. Solve $\frac{3x}{4} + \frac{5x}{6} = 38$.

The L. C. M. of 4 and 6 is 12.

Multiplying both sides of the equation by 12, we have

$$12 \times \frac{3x}{4} + 12 \times \frac{5x}{6} = 12 \times 38$$

or $9x + 10x = 12 \times 38$

or $19x = 12 \times 38$

or $x = \frac{12 \times 38}{19}$

or $x = 24$.

[Verification. Since $\frac{3 \times 24}{4} + \frac{5 \times 24}{6} = 18 + 20 = 38$, therefore the answer is correct.]

Example 2. Solve $\frac{3x}{5} + \frac{5}{6} = \frac{x}{2} + \frac{3}{4}$.

The L. C. M. of 5, 6, 2 and 4 is 60.

Multiplying both sides by 60, we have

$$36x + 50 = 30x + 45$$

$$\therefore 36x - 30x = 45 - 50$$

$$\therefore 6x = -5$$

$$\therefore x = -\frac{5}{6}.$$

[Verification. L. H. S. $= \frac{3}{5} \times (-\frac{5}{6}) + \frac{5}{6} = -\frac{1}{2} + \frac{5}{6} = \frac{1}{3}$.

R. H. S. $= \frac{1}{2} \times (-\frac{5}{6}) + \frac{3}{4} = -\frac{5}{12} + \frac{3}{4} = \frac{1}{3}$.

Thus L. H. S. = R. H. S., and the solution is correct.]

Example 3. Solve $\frac{2x+7}{5} - 6 = \frac{3x+4}{10} + 8$.

By transposition, we have

$$\frac{2x+7}{5} - \frac{3x+4}{10} = 8 - 6 = 2.$$

Multiplying both sides by 10, the L. C. M. of 5 and 10, we have

$$2(2x+7) - (3x+4) = 20.$$

Removing the brackets, we have

$$4x + 14 - 3x - 4 = 20$$

$$\text{or} \quad x + 10 = 20$$

$$\text{or} \quad x = 10.$$

[Verification. L. H. S. $= \frac{2 \times 10 + 7}{5} - 6 = 11\frac{2}{5}$.

R. H. S. $= \frac{3 \times 10 + 4}{10} + 8 = 11\frac{2}{5}$.

Thus L. H. S. = R. H. S., and the solution is correct.]

Example 4. Solve $\cdot 4x - \cdot 35 = \cdot 34x + \cdot 01$.

Expressing the decimals as vulgar fractions, we have

$$\frac{2x}{5} - \frac{7}{20} = \frac{17x}{50} + \frac{1}{100}.$$

The L.C.M. of 5, 20, 50, 100 is 100.

Multiplying both sides by 100, we have

$$40x - 35 = 34x + 1$$

$$\therefore 40x - 34x = 35 + 1$$

$$\text{or } 6x = 36$$

$$\text{and } x = 6.$$

Another method

$$\cdot 4x - \cdot 35 = \cdot 34x + \cdot 01.$$

By transposition, we have

$$\cdot 4x - \cdot 34x = \cdot 01 + \cdot 35$$

$$\text{or } \cdot 06x = \cdot 36$$

$$\text{or } x = 6.$$

EXERCISE 18.

Solve the following equations and *verify* each solution :

1. $\frac{x}{2} + \frac{x}{3} = 5.$

2. $\frac{x}{4} - \frac{x}{6} = 3.$

3. $\frac{x}{2} + \frac{2x}{3} = \frac{7}{2}.$

4. $\frac{5x}{6} - \frac{x}{4} = 14.$

5. $\frac{x}{2} - 5 = 7 - \frac{5x}{6}.$

6. $\frac{2x}{5} - \frac{3}{4} = \frac{x}{2} - \frac{5}{6}.$

7. $\frac{3}{x} = 2.$ [*Hint.* Multiply both sides by x .]

8. $\frac{1}{x} - \frac{1}{6} = \frac{1}{8}.$

9. $\frac{3x+1}{5} = \frac{2x-1}{11}.$

10. $\frac{y-3}{5} + \frac{y-5}{3} = 0.$

11. $\frac{3y-2}{7} - \frac{5y-1}{14} = 0.$

12. $\frac{x}{2} - 3 = \frac{3x}{4} - \frac{5x}{6} + 7.$

13. $\frac{6x}{5} + \frac{3x}{4} - 11 = \frac{3x}{20} - 18 + \frac{5x}{2}.$

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$$14. \quad \frac{4x-1}{5} = \frac{3x+1}{4} - 1.$$

$$15. \quad \frac{x-1}{14} + \frac{x-2}{21} = \frac{x-3}{7}.$$

$$16. \quad \frac{2x+3}{11} + 2 = \frac{2x-1}{22} + \frac{x+3}{6}.$$

$$17. \quad 3 + \frac{5x-8}{16} = \frac{x+4}{8} + 6\frac{1}{2}.$$

$$18. \quad 12 + \frac{x+8}{6} = 4 + \frac{3x-16}{8}.$$

$$19. \quad \frac{2x-1}{3} - \frac{1}{4} = x - \frac{x+1}{2}.$$

$$20. \quad .5x + 1 \cdot 2x = 3 \cdot 4.$$

$$21. \quad 1 \cdot 32x + .02x = 1 \cdot 19 + x.$$

$$22. \quad .2y + .05 = .12 + .15y.$$

$$23. \quad \frac{.24 - .5x}{6} = \frac{.3x + 1}{4}.$$

$$24. \quad \frac{.3x - 1}{5} = \frac{.6 - .2x}{3}.$$

$$25. \quad x - \frac{2x - .3}{.7} = \frac{.5 - x}{.35}.$$

$$26. \quad 1 \cdot 2x - \frac{.16x - .05}{.5} = .4x.$$

$$27. \quad \frac{x}{.5} - \frac{1}{.05} + \frac{x}{.005} - \frac{1}{.0005} = 0.$$

$$28. \quad \frac{x + .\dot{3}}{1.\dot{6}} + \frac{2x - .\dot{6}}{1.\dot{3}} = 6x + 2 \cdot 5.$$

$$29. \quad \frac{1}{2}(x-2) + \frac{1}{3}(x-3) = \frac{1}{4}(x-4).$$

$$30. \quad \frac{7x-3}{2} + 1 = \frac{3x+5}{4} + \frac{7x-2}{3}.$$

$$31. \quad \frac{2x-1}{2} + \frac{3x-4}{6} = \frac{6+x}{4} - \frac{x+1}{9}.$$

$$32. \quad \frac{5+3x}{5} - \frac{3x-1}{2} = \frac{3+5x}{3} + \frac{2x+1}{4}.$$

$$33. \quad \frac{x+2\frac{1}{2}}{15} + \frac{x+3\frac{1}{3}}{25} = \frac{x+4\frac{1}{6}}{55}.$$

$$34. \quad \frac{1}{3}(x-4) - \frac{1}{6}(2x+3) = \frac{7(5x+6)}{9} \text{ correct up to 2 decimal}$$

places.

$$35. \quad \text{What value of } x \text{ will make } \frac{x}{2} + \frac{x}{3} + \frac{x}{4} - 2 \text{ equal to } x?$$

5. Change of subject in a formula.

Let us consider the formula

$$I = \frac{PRT}{100} \quad \dots \dots \dots (i)$$

Here the subject of the formula is I and it can be determined if P , R and T be known.

If this formula be reduced to the form

$$100 \times I = PRT,$$

we can deduce from it

$$P = \frac{100 \times I}{RT} \quad \dots \dots \dots (ii)$$

$$R = \frac{100 \times I}{PT} \quad \dots \dots \dots (iii)$$

$$T = \frac{100 \times I}{PR} \quad \dots \dots \dots (iv)$$

In (ii) the subject of the formula is P and it can be determined if I , R and T be known.

In (iii) the subject of the formula is R and it can be determined if I , P and T be known.

In (iv) the subject of the formula is T and it can be determined if I , P and R be known.

The process illustrated above is called **changing the subject** of a formula.

Example 1. A rectangular room is l ft. long, b ft. broad and h ft. high. If the area of the four walls is A sq. ft., we know that $A = 2h(l + b)$.

Here A is the subject of the formula ; change the formula so that l may become the subject.

$$A = 2h(l + b)$$

\therefore

$$2h(l + b) = A.$$

Divide each side by $2h$,

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$$l + b = \frac{A}{2h}.$$

Subtract b from both sides,

$$l = \frac{A}{2h} - b.$$

EXERCISE 19.

1. If the area of a triangle whose base is b in. and height h in. is A sq. in., $A = \frac{1}{2}bh$.

Make (i) b the subject,

(ii) h the subject.

2. If c ft. is the circumference of a circle of radius r ft., $c = 2\pi r$. Make r the subject.

3. In a trapezoid a ft. and b ft. are the parallel sides and h ft. is the height. If its area is A sq. ft., $A = \frac{1}{2}(a+b)h$.

Make (i) h the subject.

(ii) $(a+b)$ the subject.

4. The radius of the base of a cylinder is r ft. and its height is h ft. If the area of its curved surface is A sq. ft., $A = 2r\pi h$. Make (i) r the subject, (ii) h the subject.

5. If the radius of a circle is r ft. and its area A sq. ft., $A = r^2\pi$. Make r the subject.

6. If e ft. is the edge of a cube and A sq. ft. the area of its six faces, $A = 6e^2$. Make e the subject.

7. If in an isosceles triangle the vertical angle is x degrees and a base angle y degrees, $x + 2y = 180$.

Make (i) x the subject.

(ii) y the subject.

8. If x be increased by $r\%$, the result $y = x \left(1 + \frac{r}{100}\right)$

Make r the subject. Find r if $x = 20$ and $y = 21$.

9. If r ft. is the radius of the base of a cylinder, h ft. its height and V cubic ft. its volume, $V = r^2\pi h$. Make (i) r the subject, (ii) h the subject.

10. Given $v = u + ft$; find (i) f in terms of u, v, t , (ii) t in terms of u, v, f .
11. Given $S = 16t^2$. Make t the subject.
12. If a ft. is one side of a right-angled triangle, b ft. the other side and c ft. its hypotenuse, $c^2 = a^2 + b^2$. Make a the subject.
13. If F degrees Fahrenheit is the same temperature as C degrees Centigrade, $F = 32 + \frac{9C}{5}$; make C the subject, and find C when $F = 50$.
14. If the radius of the base of a circular cone is r ft., its height h ft. and volume V cubic ft., $V = \frac{1}{3}\pi r^2 h$.
Make (i) h the subject, (ii) r the subject.
15. If $s = ut + \frac{1}{2}ft^2$, make f the subject.
16. The temperature t , the pressure p and the volume v of a quantity of gas are connected by the formula $\frac{pv}{t+273} = 10$.
Make t the subject, and find t if $p = 3, v = 1050$.

Symbolical Expressions

6. Algebra is largely used for the solution of problems of various kinds, but before attempting them, the student must have sufficient practice in expressing the given statements symbolically.

EXERCISE 20. (Oral)

1. What is the sum of 7 and a ?
2. What is the sum of m and n ?
3. What is the difference between q and 15?
4. What is the difference between p and q ?
5. One part of x is 7; what is the other?
6. One part of $2m$ is n ; what is the other?

7. One part of $2m+n$ is $m+n$; what is the other ?
8. By how much is m greater than 17 ?
9. By how much is m less than 20 ?
10. The sum of two numbers is m and one of them is n ; what is the other ?
11. The sum of two numbers is $2x+19$, one of them is $x+5$; what is the other ?
12. The difference between two numbers is d , the greater is 25 ; what is the other ?
13. What is the selling price if the cost price is Rs. 160 and the profit is Rs. x ?
14. What is the excess of 53 over n ?
15. What is the excess of m over 34 ?
16. What is the excess of p over q ?
17. What is the defect of 19 from 23 ?
18. What is the defect of m from 40 ?
19. What is the defect of x from y ?
20. What number is less than 60 by m ?
21. What number is less than $2m$ by n ?
22. By how much does 21 exceed n ?
23. By how much does $2p$ exceed q ?
24. If n is a factor of 28, what is the other factor ?
25. If m is a factor of n , what is the other factor ?
26. If m be divided into 5 equal parts, what is the value of each part ?
27. If the product of two numbers is $2p^2$ and one of them is p , what is the other ?
28. How many times is 8 contained in 48 ?
29. How many times is a contained in 48 ?
30. How many times is m contained in n ?
31. There are three numbers, each equal to a ; what is their product ?
32. What is five times x increased by 15 ?

33. What is seven times n diminished by 12?
34. What is three times p increased by q ?
35. What is six times m diminished by n ?
36. x is multiplied by 4 and 13 is added to the product; what is the result?
37. y is doubled and then 5 is added to it; what is the result?
38. One-fifth of m is increased by 7; what is the result?
39. One-third of x is diminished by 9; what is the result?
40. Three-fourths of y is increased by 11; what is the result?
41. Five-sixths of m is diminished by 13; what is the result?
42. Write down three consecutive numbers of which 15 is the middle one; of which x is the middle one.
43. Write down three consecutive numbers of which 11 is the least one; of which k is the least one.
44. Write down three consecutive numbers of which 27 is the greatest one; of which p is the greatest one.
45. Write down three consecutive numbers of which $2m+3$ is the middle one.
46. Write down three consecutive even numbers of which $2n+4$ is the middle one.
47. Write down three consecutive even numbers of which $2x-6$ is the least one.
48. Write down three consecutive even numbers of which $2x+2$ is the greatest one.
49. Write down three consecutive odd numbers of which $2n-3$ is the least one.
50. Write down three consecutive odd numbers of which $2m+1$ is the middle one.
51. Write down three consecutive odd numbers of which $2n+5$ is the greatest one.

If the digits of a number beginning from the left-hand side be 3 and 7, the number is 37, and its value is $10 \times 3 + 7$; when the digits of 37 are reversed, it becomes 73, and its value is $10 \times 7 + 3$.

Similarly, if the digits of a number beginning from the left-hand side be x and y , the value of the number $= 10 \times x + y = 10x + y$; when these digits are reversed, the value of the number thus formed $= 10 \times y + x = 10y + x$.

52. Represent the numbers whose digit in the tens place is 8 and the digit in the units place is n .

53. Represent the number whose digit in the tens place is k and the digit in the units place is 4.

54. If the digits of a number beginning from the left be m and n , what is its value?

55. In the preceding question, if the digits be reversed, how would you represent that number?

56. Represent the number whose digit in the tens place is $3p$ and the digit in the units place is $2q$.

57. A man is m years old now; how old was he 16 years back? n years back?

58. A man is 35 years old now; how old will he be 12 years hence? x years hence?

59. How old will a man be k years hence, if he is now y years old?

60. How old is a man now who was 17 years of age 13 years ago? x years ago?

61. How old is a man now who was p years of age 18 years ago? n years ago?

62. A man will be 60 years old in k years; how old is he now?

63. A man will be n years old in m years; how old is he now?

64. The age of a boy is $(m-5)$ years; what was his age k years ago? What will be his age p years hence?

65. A father is 37 years older than his son. The age of the son is $(m-3)$ years; what is the age of the father?
66. A man walks at the rate of x miles an hour; (i) how far does he walk in 3 hours? (ii) in h hours?
67. A man walks at the rate of x miles an hour; (i) how far does he walk in 4 hours? (ii) in t hours?
68. What is the speed per hour of a man who travels (i) 40 miles in 8 hours? (ii) 40 miles in x hours? (iii) m miles in t hours?
69. A train goes at the rate of 28 miles per hour; (i) in what time will it go 280 miles? (ii) m miles?
70. A train goes at the rate of s miles per hour; (i) in what time will it go 250 miles? (ii) m miles?
71. The cost price of an article is $3m$ rupees, and the gain is Rs. 5: find its selling price.
72. The cost price of an article is Rs. $7x$, and the loss is Rs. 9; find its selling price.
73. The selling price of an article is Rs. $2n$, and the loss is Rs. 6; find its cost price.
74. The price of 7 cows is Rs. 350; what is the price of one cow? of x cows?
75. If one cow costs k rupees; how many can be bought for r rupees?
76. What is $12\frac{1}{2}$ per cent. of 40? of x ?
77. What is m per cent. of Rs. 250? of Rs. k ?
78. A man buys goods for Rs. x , and sells them at a gain of 7 per cent.; what is the selling price?
79. A man buys goods for Rs. $3m$, and sells them at a loss of 12 per cent.; what is the selling price?
80. A man buys goods for Rs. c , and sells them at a gain of p per cent.; what is the selling price?

Solution of Problems

7. The application of equations in the solution of problems is of the utmost importance.

It is usual to take the following steps in the solution of problems:

(i) Read the problem *very carefully* and try to understand the relation between the quantities stated therein. (This step is very important.)

(ii) Represent the unknown quantity by the symbol x .

(iii) State the *conditions* of the problem in *symbolic language*, so as to obtain two expressions which are equal. (This step is very important.)

(iv) Solve the equation thus formed.

(v) Verify the solution.

Example 1. The excess of twice a certain number over 9 is 15; find the number.

Let x be the required number

\therefore twice that number $= 2x$

\therefore the excess of twice the number over 9 $= 2x - 9$.

As this excess is equal to 15,

$$\therefore 2x - 9 = 15$$

$$\therefore 2x = 24$$

$$\text{and } x = 12.$$

[Verification. Since $2 \times 12 - 9 = 15$, the answer is correct.]

Example 2. One number is greater than another by 11 and their sum is 49; find them.

Let x be the greater number,

then the smaller number $= x - 11$.

As their sum is equal to 49,

$$\therefore x + (x - 11) = 49.$$

or $2x - 11 = 49$

or $2x = 60$

$\therefore x = 30$, the greater number

and $x - 11 = 30 - 11 = 19$, the smaller number.

[Verification. Since $30 - 19 = 11$ and $30 + 19 = 49$, the answer is correct.]

Example 3. Divide Rs. 64 between A and B , so that three times A 's share may be greater than four times B 's share by Rs. 10.

Let x be the number of rupees in A 's share,

then $(64 - x)$ = the number of rupees in B 's share.

\therefore three times A 's share $= 3x$

and four times B 's share $= 4(64 - x)$.

According to the condition, $3x$ is greater than $4(64 - x)$ by 10.

$\therefore 3x = 4(64 - x) + 10$

or $3x = 256 - 4x + 10$

or $3x = 266 - 4x$

$\therefore 7x = 266$

and $x = 38$.

$\therefore 64 - x = 64 - 38 = 26$.

$\therefore A$'s share = Rs. 38 and B 's share = Rs. 26.

[Verification. Since 3×38 is greater than 4×26 by 10 and $38 + 26 = 64$, the answer is correct.]

Example 4. I thought of a number, doubled it, then added 7, the result was 31; what was the number thought of?

Let x be the number thought of, then the successive steps would be:

$x, 2x, 2x + 7,$

$\therefore 2x + 7 = 31$

$\therefore 2x = 24$

$\therefore x = 12$.

[Verification $\therefore 12 \times 2 + 7 = 31$, \therefore the answer is correct.]

EXERCISE 21.

Solve the following problems and *verify* the answer in each case :

1. The defect of an angle from 90° is 15° ; find the angle.
2. The excess of an angle over a right angle is 12° ; find the angle.
3. In an isosceles triangle, the vertical angle is one-half of each of the base angles. Find each angle.
4. One angle is one-fifth of another, and their difference is 20° ; find them.
5. A is twice as old as B , and the sum of their ages is 72 years; find their ages.
6. A is three times as old as B , and the difference of their ages is 30 years; find their ages.
7. One angle is greater than another by 32° , and their sum is 106° ; find them.
8. The sum of two numbers is 83, and one of them exceeds the other by 17; find them.
9. If 28 be added to five times a number, the result is 78; find the number.
10. If 7 be taken away from four times a number, the result is 69; find the number.
11. If 9 be added to one-third of a number, the result is 19; find the number.
12. If 11 be taken away from one-fifth of a number, the result is 23; find the number.
13. Think of a number, double it, and add 15. If the result is 71, what was the number thought of?
14. Think of a number, divide it by 4, and add 9 to it; the result is 15. What was the number thought of?
15. Think of a number, take away 5 from it, multiply the result by 4, and add 10 to it; it becomes equal to 90. Find the original number.

16. Divide Rs. 96 between A and B , so that four times A 's share may be greater than three times B 's share by Rs. 13.

17. Divide Rs. 150 between A and B , so that five times A 's share may be less than six times B 's share by 9.

18. A has x apples, B has $x+5$, and C has $x-4$, and they together have 49 apples ; find what each has got.

19. Divide Rs. 81 among A , B and C , so that B may get Rs. 7 more than A , and C 6 less than twice A 's share.

20. Divide Rs. 340 among A , B and C , so that B may get Rs. 30 more than A and C Rs. 50 less than twice A 's share.

21. Divide Rs. 500 among A , B and C , so that B may get Rs. 60 more than one-half of A 's share and C may get Rs. 90 more than one-fourth of A 's share.

22. The sum of the interior angles of a figure of n sides is always $(2n-4)$ right angles. If the sum of the interior angles of a figure is 16 right angles, how many sides has it?

Example 5. Divide 180 into two parts, so that one-third of one part may be greater than one-fourth of the other by 18.

Let x be one of the parts ; then $180-x$ would be the other part.

$$\text{One-third of the first part} = \frac{x}{3}.$$

$$\text{One-fourth of the second part} = \frac{1}{4}(180-x).$$

According to the condition given in the problem, we have

$$\frac{x}{3} = \frac{1}{4}(180-x) + 18.$$

Multiplying both sides by 12, we have

$$4x = 3(180-x) + 216$$

$$\text{or} \quad 4x = 540 - 3x + 216$$

$$\text{or} \quad 7x = 756$$

$x = 108$, the first part,

and $(180 - x) = 180 - 108 = 72$, the second part.

[Verification. Since $\frac{1}{3} \times 108 = 36$, and $\frac{1}{4}$ of $72 = 18$, also 36 is greater than 18 by 18, therefore the solution is correct.]

23. What is the number whose one-half, one-third, and one-fourth added together give us 65?

24. What is the number whose sixth part exceeds its seventh part by 10?

25. What is the number whose fourth part falls short of its third part by 5?

26. Divide 81 into two parts, so that five-sixths of the smaller may exceed seven-fifteenths of the larger by 9.

27. A post is one-sixth of its length in mud, one-fourth of its length in water, and 12 ft. above the water; find its length.

28. A man goes one-half of his journey by a railway train, one-fourth of it by a motor-car, one-sixth of it on a horse, and the remaining 10 miles on foot; find the total length of the journey.

29. Out of a cask of wine one-seventh had leaked away, 6 gallons were sold, and then the cask was three-fourths full; find the capacity of the cask.

30. A man leaves one-fourth of his property to his wife, one-sixth of it to each of his two sons, one-eighth of it to each of his two daughters, and the rest, amounting to Rs. 1,200 for a school; find his total property.

31. Find a number whose one-half added to 12 exceeds the sum of its third and fourth parts by 5.

32. One-sixth of a number is taken away from one-half of it; the result is less than the sum of its fourth and eighth parts by 15. Find the number.

33. To a certain number I add one-third of it; the result is as much above 100 as the number is below 103. Find the number.

34. Divide Rs. 860 between A and B so that five-sevenths of A 's share may be less than three-fourths of B 's share by Rs. 30.

35. The difference between two numbers is 16, and one-sixth of one is less than one-fourth of the other by 5; find the numbers.

36. The sum of two numbers is 678. When the greater is divided by the less, the quotient is 5 and the remainder is 6. Find the numbers.

[Hint. Dividend = quotient \times divisor + remainder.]

37. The sum of two numbers is 1064. When the greater is divided by the less, the quotient is 4 and the remainder is 14. Find the numbers.

38. The difference between two numbers is 431. When the greater is divided by the less, the quotient is 3 and the remainder is also 3. Find the numbers.

Example 6. A is 24 years older than B ; 10 years back A 's age was five times the age of B ; find their ages.

Let x years be the age of A ,
 then the age of B is $(x - 24)$ years.
 10 years back, the age of A was $(x - 10)$
 years and that of B was $\{ (x - 24) - 10 \}$
 years.

According to the condition given in the problem we have

$$x - 10 = 5 \{ (x - 24) - 10 \}$$

$$\therefore x - 10 = 5(x - 34)$$

$$\therefore x - 10 = 5x - 170$$

$$\therefore 4x = 160$$

$$\therefore x = 40, \text{ the age of } A,$$

and $x - 24 = 40 - 24 = 16$, the age of B .

[Verification. Since $40 - 16 = 24$ and $(40 - 10)$ or 30 is five times $(16 - 10)$ or 6, therefore the answer is correct.]

39. A 's age is three-eighths of B 's age ; the difference of their ages is 15 ; find their ages.

40. A father is 20 years older than his son ; 12 years back the age of the father was six times that of his son ; find their ages.

41. A father is 24 years older than his son ; 15 years hence the father will be $\frac{5}{3}$ times as old as his son ; find their ages.

42. What is A 's present age if he is now three times as old as B , and was 5 times as old as B 8 years back ?

43. A father is twice as old as his son ; 14 years ago he was three times as old as his son ; find their ages.

44. A is 10 years older than B ; 12 years ago five-sevenths of A 's age exceeded five-eighths of B 's age by 10 ; find their ages.

45. A father is 26 years older than his son ; 16 years hence three-fifths of the father's age will exceed two-thirds of his son's age by 14 ; find their ages.

Example 7. Find three consecutive even numbers whose sum is 96.

Let $2n$ be the middle one of the consecutive even numbers,

then $2n - 2$ = the least one of the consecutive even numbers,

and $2n + 2$ = the greatest one of the consecutive even numbers.

Since their sum is given equal to 96,

$$\therefore (2n - 2) + 2n + (2n + 2) = 96$$

$$\text{or} \quad 6n = 96$$

$$\therefore n = 16.$$

$$\therefore 2n - 2 = 30, \quad \text{the least number.}$$

$$2n = 32, \quad \text{the middle number,}$$

$$\text{and } 2n + 2 = 34, \quad \text{the greatest number.}$$

Example 8. Find two consecutive odd numbers so that one-third of the greater may exceed one-fifth of the less by 8.

Let $2n+1$ be the greater of the consecutive odd numbers,
 then $2n-1 =$ „ lesser „ „ „ „

According to the condition given in the problem, we have

$$\frac{1}{3}(2n+1) = \frac{1}{5}(2n-1) + 8$$

$$\text{or } 5(2n+1) = 3(2n-1) + 120$$

$$\text{or } 10n+5 = 6n-3+120$$

$$\text{or } 4n = 112$$

$$\text{or } n = 28.$$

$\therefore 2n+1 = 57$, the greater consecutive odd number,
 and $2n-1 = 55$, „ lesser „ „ „

46. Find three consecutive numbers whose sum is 87.
47. Find four consecutive numbers whose sum is 74.
48. Find three consecutive even numbers whose sum is 48.
49. Find three consecutive odd numbers whose sum is 81.
50. Find two consecutive numbers such that one-third of the greater exceeds one-fifth of the less by 7.
51. Find two consecutive even numbers such that one-eighth of the greater may exceed one-tenth of the less by 3.
52. Find two consecutive odd numbers such that one-seventh of the greater may fall short of one-fifth of the less by 4.
53. Find two consecutive numbers such that seven-eighths of the greater may exceed five-sixths of the less by 6.
54. Find two consecutive odd numbers such that two-thirds of the greater may exceed three-fifths of the less by 9.

Example 9 There are two digits in a number, the digit in the units place being 8. If 9 be added to the number, the order of the digits is reversed; find the number.

Let $x =$ the digit in the tens place.

$\therefore 8 =$ the digit in the units place,

\therefore the value of the number $= 10 \times x + 8$.

When the order of the digits is reversed, the value of the new number $= 10 \times 8 + x$.

According to the condition given in the problem, we have

$$(10x + 8) + 9 = 10 \times 8 + x$$

or $10x + 17 = 80 + x$

or $9x = 63$

or $x = 7$.

Thus the required number $= 78$.

[**Verification.** Since $78 + 9 = 87$, and we get 87 by reversing the digits of 78, therefore the answer is correct.]

55. A number consists of two digits, the sum of the digits is 12. If 36 be added to the number, the digits are reversed. Find the number.

56. A number consists of two digits, the digit in the units place is 5. If 27 be added to it, the digits are reversed. Find the number.

57. A number consists of two digits whose sum is 8. If 54 be subtracted from it, the digits are reversed. Find the number.

58. A number consists of two digits, the digit in the tens place being 5. If 18 be subtracted from it, the digits are reversed. Find the number.

59. A number consists of two digits, the digit in the units place is one-third of the digit in the tens place. If the digits be reversed, the new number falls short of the original by 36. Find the number.

60. A number consists of two digits, the digit in the units place is one-half of the digit in the tens place. If 9 be subtracted from the number, the digits are reversed. Find the number.

61. Reverse the digits of a number, it will be three-

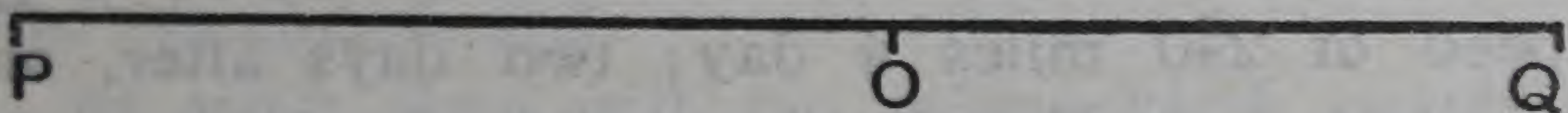
eighths of what it was before ; also the difference between the digits is 5. Find the number.

62. There is a number of two digits whose difference is 2. If it be increased by three times the sum of its digits, the digits are reversed. Find the number.

63. A number consists of two digits ; the digit in the units place is three times the digit in the tens place. If the number be divided by the sum of the digits, the quotient is 3 and the remainder is 3. Find the number.

Example 10. Two trains with speeds 32 miles and 24 miles per hour respectively, starting at the same time, run in opposite directions between two stations P and Q , 336 miles apart one going from P to Q and the other from Q to P ; find when and where they will meet.

Suppose that the two trains meet at O after x hours, then



the number of miles travelled by the first train in x hours = $32x$ and the number of miles travelled by the second train in x hours = $24x$.

The sum of these distances is equal to PQ . or 336 miles.

$$\therefore 32x + 24x = 336$$

$$\text{or } 56x = 336$$

$$\text{or } x = 6.$$

Hence the two trains meet after 6 hours and at a distance of 6×32 , or 192 miles from P .

Example 11. A and B are travelling on the same road in the same direction, A with the speed of 5 miles an hour and B in a motor-car with the speed of 60 miles an hour. A is 20 miles ahead of B . Find when and where B will overtake A .

Suppose B overtakes A after x hours; then the number of miles travelled by A in x hours = $5x$ and the number of miles travelled by B in x hours = $60x$.

According to the condition given in the problem, we have

$$60x - 20 = 5x$$

or

$$55x = 20$$

or

$$x = \frac{4}{11}.$$

Hence B overtakes A after $\frac{4}{11}$ hour or $21\frac{9}{11}$ minutes, and at a distance of $\frac{4}{11} \times 60$, or $21\frac{9}{11}$ miles, from his own starting-point.

64. A train running at the rate of 24 miles an hour leaves Amritsar for Jullundur at 7 p. m., and another train leaves Jullundur for Amritsar at 8 p. m. at the rate of 32 miles an hour. Find the time when the two trains meet, the distance between Amritsar and Jullundur being 50 miles.

65. There are two ports P and Q , the distance between them being 5625 miles. A steamer sails from P to Q with the speed of 240 miles a day; two days after, another steamer starts from Q and sails towards P with the speed of 250 miles a day. Find when and where they meet.

66. A sets out for a walk at the rate of $3\frac{1}{2}$ miles an hour; 3 hours afterwards B cycles after him at the rate of 9 miles an hour. When will B overtake A ?

67. A and B travel in opposite directions from two towns, 135 miles apart, and meet in 15 hours; if A goes twice as fast as B , what will be the speed of B ?

68. A train with the speed of 32 miles an hour starts from a station $\frac{1}{2}$ hour after a goods train, and overtakes it in 50 minutes. Find the speed of the goods train.

69. A man walked to the top of a hill at the rate of $2\frac{1}{2}$ miles an hour and down the hill at the rate of $3\frac{1}{2}$ miles an hour. The journey took him 4 hours in all; find the length of his journey.

70. A man has $7\frac{1}{3}$ hours at his disposal; how far can he go in a motor-car, running with the speed of 36 miles

an hour so as to return home in time in a tonga running with the speed of 8 miles an hour?

71. A and B start at the same time from two towns, 39 miles apart, and meet in 4 hours and 20 minutes. If A walks 1 mile an hour faster than B , find where he meets B .

72. A train takes 8 hours to travel from P to Q . Had it gone 2 miles an hour faster, it would have taken $\frac{1}{2}$ hour less to reach Q . Find its speed and the distance between P and Q .

73. A and B travel along the same road, A on foot with the speed of 5 miles an hour, and B in a motor-car with the speed of 60 miles an hour. A is 10 miles ahead of B ; find when and where B will overtake A .

74. In the last question, find when B will be ahead of A by (i) 22 miles, (ii) 15 miles, (iii) 30 miles.

[Hint. In the next three questions proceed as in the last two questions.]

75. At what time will the minute-hand overtake the hour-hand (i) between 2 and 3? (ii) between 4 and 5?

76. At what time will the minute-hand be ahead of the hour-hand by 30 minute-divisions (or opposite) (i) between 4 and 5? (ii) between 2 and 3?

77. At what time will the minute-hand be ahead of the hour-hand by 15 minute-divisions (or at right angles) (i) between 5 and 6? (ii) between 6 and 7?

Example 12. A person invested Rs. 440 in buying sheep and goats at the rate of Rs. 6 and Rs. 8 each respectively. If the total number of sheep and goats be 65, find the number of sheep and the number of goats.

Let the number of sheep purchased be x ,
 then the number of goats purchased = $65 - x$,
 \therefore the price of the sheep = Rs. $6x$,
 and the price of the goats = Rs. $8(65 - x)$.

According to the condition given in the problem, we have

$$6x + 8(65 - x) = 440$$

or $6x + 520 - 8x = 440$

or $2x = 80$

or $x = 40$ (the number of sheep),

and $65 - x = 65 - 40 = 25$ (the number of goats).

[**Verification.** The price of the sheep $= 40 \times 6$ rupees = Rs. 240; the price of the goats $= 25 \times 8$ rupees = Rs. 200. The sum of the prices = Rs. 240 + Rs. 200 = Rs. 440, therefore the answer is correct.]

Example 13. The population of a town is 15,000; the annual death-rate of males is $4\frac{1}{2}\%$ and that of females is $5\frac{1}{2}\%$. The total number of deaths during the year is 745; find the number of males and females in the town.

Let the number of males in the town $= x$,
then the number of females in the town $= 15000 - x$.

Annual number of male deaths

$$= x \times \frac{4\frac{1}{2}}{100} = \frac{9x}{200}.$$

Annual number of female deaths

$$= (15000 - x) \times \frac{5\frac{1}{2}}{100} = \frac{11(15000 - x)}{200}.$$

According to the condition given in the problem, we have

$$\frac{9x}{200} + \frac{11(15000 - x)}{200} = 745.$$

Multiplying both sides by 200, we have

$$9x + 11(15000 - x) = 149000$$

or $9x + 165000 - 11x = 149000$

or $2x = 16000$

or $x = 8000$, (the number of males),

$15000 - x = 15000 - 8000 = 7000$, (the number of females).

The students should verify the answer.

78. Two pieces of cloth, measuring 80 yds., cost together Rs. 57. One piece was bought at 9a. per yd. and the other at 13a. per yd. Find the length of each.

79. If 8 pounds of tea and 12 pounds of coffee together cost Rs. 14 8 as. and a pound of tea cost 4 as. more than a pound of coffee, find the cost of each per pound.

80. A person bought a number of mangoes for Rs. 2 10 as. and found that 18 of them cost as much under 8 annas as 27 of them cost over 7 annas. Find the number of mangoes bought.

81. A person bought sugar of two different kinds, and paid in all Rs. 13 12 as., the better one at the rate of 5 as. per seer and the worse at 4 as. per seer; the total amount of both kinds of sugar was 50 seers. Find the number of seers of each kind.

82. A person had Rs. 9,900, a part of which he lent out at 8% and the rest at 6%; the interest amounted to Rs. 702. How much was lent out at 8%?

83. A sum of Rs. 3,200 was lent out at simple interest, partly at 5% and partly at $6\frac{1}{2}\%$ per annum. The total annual interest amounted to Rs. 190. How much was lent out at each rate?

84. A person had a capital of Rs. 4,600, a part of which he invested in a business yielding 10% annual profit, and the rest he lent out at 8% per annum; the whole annual income amounted to Rs. 416. Find how much he invested in the business.

85. The population of a town is 6,600; the annual death-rate of males is $3\frac{1}{2}\%$ and that of females is $3\frac{3}{4}\%$; the total number of deaths during the year is 239. Find the number of males and females in the town.

*86. A tradesman bought an equal number of goats of two kinds, one at £1 4s. each and the other at £1 10s. each. If he had spent his money equally on the two kinds, he would have had one goat more. How many did he buy of each kind?

Example 14. An army of 8,000 soldiers is formed into a solid square, and 79 men are left over. Find the number of men in the front row.

Let x be the number of men in the front row of the solid square,

then x^2 is the number of men in the solid square.

According to the condition given in the problem, we have

$$x^2 + 79 = 8000.$$

Let the student solve it further and verify the answer.

Example 15. The length of a room exceeds its breadth by 7 ft. If the length be increased by 5 ft. and the breadth be diminished by 3 ft., the area remains unaltered. Find the dimensions of the room.

Let x be the number of ft. in the breadth,

then $(x+7)$ is the number of ft. in the length,

and $x(x+7)$ is the number of sq. ft. in the area in the first case.

The number of ft. in the length in the second case $= (x+7) + 5$,

The number of ft. in the breadth in the second case $= (x-3)$

and the number of sq. ft. in the area in the

second case $= (x-3) \{ (x+7) + 5 \}$.

According to the condition given in the problem, we have

$$x(x+7) = (x-3) \{ (x+7) + 5 \}.$$

Let the student solve it further and verify the answer.

Example 16. Divide 36 into 4 parts such that if the first be increased by 2, the second be diminished by 2, the third be multiplied by 2, and the fourth divided by 2, the result in each case is the same.

Let x be the result we get in each case.

First part $+ 2 = x$, \therefore first part $= x - 2$.

Second part $-2 = x$, \therefore second part $= x + 2$.

Third part $\times 2 = x$, \therefore third part $= \frac{x}{2}$.

Fourth part $\div 2 = x$, \therefore fourth part $= 2x$.

Since the sum of the parts is equal to the whole,

$$\therefore (x-2) + (x+2) + \frac{x}{2} + 2x = 36$$

$$\text{or } \frac{9x}{2} = 36$$

$$\text{or } x = 8.$$

Hence the first part $= x - 2 = 8 - 2 = 6$.

the second part $= x + 2 = 8 + 2 = 10$.

the third part $= \frac{x}{2} = \frac{8}{2} = 4$.

and the fourth part $= 2x = 2 \times 8 = 16$.

Example 17. A hare, 70 of her leaps before a greyhound, takes 5 leaps for every 4 leaps of the greyhound, but 2 leaps of the greyhound cover as much ground as 3 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

Let $4x$ be the number of leaps the greyhound takes to catch the hare.

then the number of leaps the hare takes in the same time $= 5x$.

Suppose the greyhound is at G and the hare at H , and the greyhound catches the hare at C .



Evidently $GC = 4x$ leaps of the greyhound

$$= \frac{3}{2} \times 4x \text{ leaps of the hare}$$

$$= 6x \text{ leaps of the hare.}$$

Again, $GC = GH + HC$

$$= (70 + 5x) \text{ leaps of the hare.}$$

$$\therefore 70 + 5x = 6x$$

$$\text{or } x = 70.$$

Hence the number of leaps the greyhound takes to catch the hare $= 4x$, or 280.

87. An army of 2,000 soldiers is formed into a solid square, and 64 men are left over; find the number of men in the front row.

88. Find the length of the side of a square whose area is exactly equal to the area of a rectangle whose sides are 225 ft. and 289 ft.

89. The length of a rectangular room exceeds its breadth by 9 ft. If the length be increased by 7 ft. and the breadth diminished by 5 ft., the area remains the same. Find the dimensions of the room.

90. The length of a rectangular room exceeds its breadth by 6 ft. If the length be increased by 9 ft. and the breadth diminished by 3 ft. the area is increased by 15 sq. ft. Find its dimensions.

91. The length of a room exceeds its breadth by 12 ft. If the length be diminished by 6 ft. and the breadth increased by 2 ft., the area is diminished by 48 sq. ft. Find its dimensions.

92. A bag contains Rs. 325 in rupees and 8-anna bits. If the amount of the latter be less than that of the former by Rs. 15, how many coins are there of each kind?

93. A purse contains three times as many 4-anna bits as 2-anna bits. The value of the coins is Rs. 17 8a. How many coins are there of each kind?

94. Divide 180 into four parts such that if the first part be increased by 5, the second diminished by 5, the third multiplied by 5 and the fourth divided by 5, the result in each case is the same.

95. Divide 155 into four parts such that if the first part be diminished by 4, the second be increased by 5, the third be divided by 2, and the fourth be multiplied by 3, the result is the same.

96. Divide 81 into four parts such that if the first part be diminished by 5, the second be increased by 1, the third be multiplied by 2, and the fourth be divided by 3, the result is the same.

*97. A certain sum of money is to be distributed amongst a number of boys; on calculation it is found that if Rs. 4 be given to each, Rs. 13 would be left over, and if Rs. 5 be given to each, Rs. 15 would be wanting. Find the number of boys and the sum.

*98. A person has h hours at his disposal; he intends to walk a distance of d miles during this period. He finds that if he walk 3 miles an hour, he would require 40 minutes more to complete his journey, but if he walk 4 miles an hour, he would save 45 minutes. Find the values of h and d .

*99. A hare, 30 of her leaps before a greyhound, takes 4 leaps for every 3 leaps of the greyhound, but 2 leaps of the greyhound cover as much ground as 3 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

*100. A hare, 22 of her leaps before a greyhound, takes 5 leaps for every 4 leaps of the greyhound, but 5 leaps of the greyhound cover as much ground as 9 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

SECTIONAL REVISION I

PAPER I

1. Use squared paper to show that
(i) $8 - 3 - 12 = -7$. (ii) $5a - 8a + a = -2a$.
2. Find the value of $2x^3 - 3x^2 + x - 5$ when $x = -2, -1, 0$.
3. Divide $x^4 - x^2y^2 - x^2z^2 + y^2z^2$ by $x^2 - xy + xz - yz$.
4. Prove by different methods that $6a - 2(8a - 3a) \equiv -4a$.
5. Solve the equation $(3x + 20) - (x - 3x + 2) = 21$.
6. A number consists of two digits whose difference is 4. If it be increased by three times the sum of its digits, the digits are reversed; find the number.

PAPER 2

1. Subtract $b \{ a - (b + c) \}$ from the sum of $a \{ a - (c - b) \}$ and $c \{ a - (b - c) \}$.
2. What must be added to $8x^3 - 12x^2 - 16x + 21$ to make it exactly divisible by $2x - 3$?
3. Simplify $\frac{a}{6} - \left[\frac{3b - 4a}{6} - \left\{ b - \frac{a - 3b}{6} - \left(3b - \frac{4a + b}{3} \right) \right\} \right]$.
4. Find the continued product of
 $(a - b), (a^2 + ab + b^2), (a^3 + b^3)$.
5. Solve the equation $\frac{5}{3}(x - 4) - \frac{7}{3}(2x - 6) = 20$.
6. A father is 21 years older than his son; 12 years hence three-fourths of the father's age will exceed two-thirds of the son's age by 18 years; find their ages.

PAPER 3

1. A person walks $3a - 2b + c$ miles east, he then walks $2a - b - 3c$ miles west, again walks $5a - b - 4c$ miles east. How far is he then from the starting-point?

2. Use squared paper to illustrate the following :
 (i) $5a + 4a - 7a = 2a$, (ii) $3a - 9a + 4a = -2a$.
3. Divide $1+x$ by $1-2x+2x^2$ up to four terms.
4. If $x=2a-3b$ and $y=3a-2b$, find the value of $(2x-3y)(x-2y)$.
5. Solve $\frac{5}{6}(x-2) - \frac{3}{8}(2x+1) + 2 = 0$.
6. A and B are travelling on the same road in the same direction, A with the speed of 4 miles an hour and B with the speed of $4\frac{3}{4}$ miles an hour. B gives A a start of 40 minutes; find when and where B will overtake A .

PAPER 4

1. Simplify $3a - [a - b - 2 \{ a - b + c - (a - b - c - a) \} + a]$.
2. The product of two expressions is $x^4 + 4x^3 + 3x^2 + 2x - 1$, and one of them is $x^2 + x + 1$; find the other.
3. Multiply $x^{2n} + x^n + 1$ by $x^n - 1$.
4. Find the value of $\frac{2a-b}{a+b} + \frac{2b-c}{b+c} + \frac{2c-a}{c+a}$ when $a=2$, $b=a+2$ and $c=a-2$.
5. Prove diagrammatically :
 (i) $(a+b)c = ac + bc$, (ii) $(a-b)c = ac - bc$,
 (iii) $(a+b)(c+d) = ac + ad + bc + bd$.
6. Solve the equation $\frac{5(3-2x)}{4} = 3(x+1) - \frac{3x+8}{6}$.

PAPER 5

1. Distinguish between (i) an index and a power, (ii) an identical equation and a conditional equation.
2. Find the value of
 (i) $(+1)^{2n}$, (ii) $(+1)^{2n-1}$, (iii) $(-1)^{2n}$, (iv) $(-1)^{2n-1}$.
3. Divide 1 by $1+a+a^2$ up to four terms.
4. Find the co-efficient of x^3 in the product of $3x^2 - 2x - 1$ and $2x^2 + 3x + 1$.
5. Find the continued product of $(a^2 + a + 1)$, $(a^2 - a + 1)$ and $(a^4 - a^2 + 1)$.

6. A cycles at the rate of 15 miles an hour from P to Q and returns at the rate of 12 miles an hour. B cycles both ways at the rate of 14 miles an hour, and takes 20 minutes less than A . Find the distance between P and Q

PAPER 6

1. Find the co-efficient of x in the product of $(x-a)$, $(x-2b)$, $(x-2c)$.
2. Multiply $x^3 - 12x^2 + 5x - 7$ by $x^4 - 3x^3 + 6x^2 - 4x + 8$, and arrange the product in ascending powers of x .
3. If $x = am^2 + 5bm + 5c$ and $y = am^2 - 6bm - 6c$, find the value of $6x + 5y$.
4. Simplify $(x-a)(x-b)(x-c)$
 $- [bc(x-a) - \{ (a+b+c)x - a(b+c) \} x]$.
5. Solve the equation $(x-3)(x-5) + 1 = (x+1)(x-2) + 4$.
6. A travels from P to Q , a distance of 60 miles, and back again, at the rate of 12 miles an hour. On his way back he meets B who travels at the rate of 8 miles an hour and who started at the same time from P . Find the distance from P of the place where they meet.

PAPER 7

1. (i) From the formula $D = d \times Q + R$, find the value of R when $D = 895$, $d = 25$, $Q = 35$.
 (ii) One factor of $27a^4 + 11a - 10$ is $3a^2 - a + 2$, find the other.
2. Add $a - 3b + 4c$, $3a + 2b - 5c$, $4a - 7b + 3c$, and multiply the result by the difference between $9a + 6c$ and $10a - b - 5c$.
3. Simplify $(5x + 4y + 1)(5x + 4y - 1) + (3x - 5y)(3x + 5y)$
 $- (4x - 3y)(4x + 3y) + 1$.
4. What must be added to $a^3 - 5a(a-3) - 1$ to make it equal to $a^3 + 5a(a+3) + 1$?
5. Solve the equation $(x+6)^2 - (5-x)^2 = 18x + 7$.
6. A person invested Rs. 5,800 at simple interest, partly at 4% and partly at 5% per annum. The total annual interest was Rs. 267; how much was lent at 4%?

PAPER 8

1. Prove by different methods that

$$x(x-1)(x-2)(x-3)+1 \equiv (x^2-3x+1)^2.$$
2. If $x=m(p+q)$ and $y=n(p-q)$, find the value of

$$\frac{x}{m} + \frac{y}{n} \text{ and } \frac{x}{m} - \frac{y}{n}.$$
3. Find the dividend when the quotient is a^2+a+2 , the divisor is a^2-3a+4 , and the remainder is $7a+3$.
4. A man walks 6 miles North, then 10 miles South, then again 8 miles North. How far is he from the starting-point? Illustrate graphically.
5. Solve the equation $\frac{.3x-.2}{5} + 1 = \frac{.6-.5x}{3} - 1.$
6. A farmer bought an equal number of sheep of two kinds, one at £1. 10s. each and the other at £2 each. If he had spent his money equally on the two kinds, he would have had one sheep more. How many of each kind did he buy?

PAPER 9

1. Simplify $3x+5-7x^2+4x-3x^2-4x^5-x^4+x^3+3x^4+6$, and arrange the result in ascending powers of x .
2. Find the value of $x(x+1)(x+2)(x+3)+1$ when $x=1, 2, 3, 4 \dots \&c.$ Show that in each case the result is a perfect square.
3. In the formula $V=\frac{1}{3}\pi r^2 h$, V is the subject. Make (i) r , (ii) h the subject of the formula.
4. Find without actual multiplication the term containing x^2 in the product of $(2x^3-x^2+x+4)$, $(3x^2+2x+1)$.
5. Solve the equation $\frac{.5x}{2} - \frac{.2x}{3} = \frac{.02x-.9}{6}.$
6. The length of a room exceeds its breadth by 11 ft. If its length be increased by 5 ft. and the breadth diminished by 4 ft., the area is diminished by 44 sq. ft.; find the dimensions of the room.

PAPER 10

1. (i) What is the square of $\frac{3}{2} a^2 b^3 c^4 d$?
 (ii) What is the square root of $\frac{1}{2} \frac{6}{5} a^4 b^{10} c^2 d^4$?
2. In India $m\%$ of the population are men and $n\%$ of them are married. What per cent of the population are the married men ?
3. Find the value of (i) $\sqrt{s(s-a)(s-b)(s-c)}$.
 (ii) $\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$
 when $a=12$, $b=5$, $c=13$ and $s=\frac{1}{2}(a+b+c)$.
4. Subtract the sum of $4(2+x)-3(x^2-1)$ and $2(x^2-1)+4(x^2-1)$ from the sum of $x^3-(1-x+x^2)$ and $2x^2(x+1)$.
5. (i) Simplify $2x - [y + \{z - (x - y + z)\}]$.
 (ii) For what value of k will $3a^4 + 5a^3 - 2a^2 - a - 1$ be equal to $3k^4 - 5k^3 - 2k^2 + k - 1$?
6. A hare, 24 of her leaps before a greyhound, takes 6 leaps for every 5 leaps of the greyhound but 7 leaps of the greyhound cover as much ground as 9 leaps of the hare. Find the number of leaps the greyhound must take to catch the hare.

CHAPTER IV

FORMULÆ

1. We add here a few important results, known as **multiplication formulae**, which should be thoroughly learnt by heart.

Formula 1. $(a + b)^2 = a^2 + 2ab + b^2$.

Proof.

$$\begin{array}{r} a + b \\ a + b \\ \hline a^2 + ab \\ + ab + b^2 \\ \hline a^2 + 2ab + b^2. \end{array}$$

Statement of the formula in words :

The square of the sum of two quantities is equal to the sum of their squares plus twice their product.

Cor. $a^2 + b^2 = (a + b)^2 - 2ab$.

Example 1. Find square of $(3x + 4y)$,

$(3x + 4y)^2$ is in form similar to $(a + b)^2$ having $3x$ in place of a and $4y$ in place of b .

$$\begin{aligned} \text{Thus } (3x + 4y)^2 &= \{ (3x) + (4y) \}^2 = (3x)^2 + 2(3x)(4y) + (4y)^2 \\ &= 9x^2 + 24xy + 16y^2. \end{aligned}$$

Example 2. Find the missing term in the perfect square

$$49a^2 + 70ab + ().$$

$$49a^2 = (7a)^2 \text{ and}$$

$$70ab = 2 \times 7a \times 5b.$$

\therefore the 2nd quantity $= 5b$.

\therefore the missing term $= (5b)^2 = 25b^2$.

Example 3. Find algebraically the value of $(309)^2$.

$$\begin{aligned} (309)^2 &= (300 + 9)^2 = (300)^2 + 2(300)(9) + (9)^2 \\ &= 90,000 + 5,400 + 81 = 95,481. \end{aligned}$$

EXERCISE 22.

Write down the square of :

1. $2x+5y$. 2. $3a+7b$. 3. $5x+3$. 4. $3+4p$.
 5. $3ab+4b^2$. 6. $8a^2+3ab$. 7. $9ax+4y^2$. 8. m^2+n^2 .
 9. $px+qy$. 10. p^3+pq^2 . 11. $2a^2b+3ab^2$. 12. $a+(-b)$.

Find the missing terms in the following perfect squares :

13. $16x^2+(\quad)+49y^2$. 14. $36m^2+60mn+(\quad)$.
 15. $(\quad)+112pq+64p^2$. 16. $121a^2+(\quad)+81$.
 17. $144m^2+72m+(\quad)$. 18. $(\quad)+104p+169$.

Find algebraically the value of :

19. $(302)^2$. 20. $(405)^2$. 21. $(703)^2$.
 22. $(806)^2$. 23. $(611)^2$. 24. $(1001)^2$.

Example 4. Simplify $(2x+3y)^2-(x+2y)^2$.

$$\begin{aligned}\text{The given expression} &= (4x^2+12xy+9y^2)-(x^2+4xy+4y^2) \\ &= 3x^2+8xy+5y^2.\end{aligned}$$

Simplify :

25. $(a+2b)^2+(2a+b)^2$. 26. $(3x+2y)^2-(2x+3y)^2$.
 27. $(px+qy)^2-(qx+py)^2$. 28. $(mx+n)^2-(mx+k)^2$.

Formula II. $(a-b)^2=a^2-2ab+b^2$.

Proof.

$$\begin{array}{r}a-b \\ a-b \\ \hline a^2-ab \\ -ab+b^2 \\ \hline a^2-2ab+b^2.\end{array}$$

Statement of the formula in words:

The square of the difference of two quantities is equal to the sum of their squares minus twice their product.

Cor. $a^2+b^2=(a-b)^2+2ab$.

Example 5. Find the square of $(3x-4y)$.

$$\begin{aligned}\text{Reducing the expression to the standard form, we have} \\ (3x-4y)^2 &= \{ (3x)-(4y) \}^2 = (3x)^2-2(3x)(4y)+(4y)^2 \\ &= 9x^2-24xy+16y^2.\end{aligned}$$

Example 6. Find the missing term in the perfect square $(\quad)-60mn+36n^2$.

$$36n^2 = (6n)^2$$

$$-60mn = -2 \times 6n \times 5m.$$

$$\therefore \text{the 2nd quantity} = 5m.$$

$$\therefore \text{the missing term} = (5m)^2 = 25m^2.$$

Example 7. Find the square of 395 by applying the formula

$$(a-b)^2 = a^2 - 2ab + b^2.$$

$$(395)^2 = (400 - 5)^2 = (400)^2 - 2(400)(5) + (5)^2$$

$$= 160,000 - 4,000 + 25$$

$$= 156,025.$$

Write down the square of :

29. $3a - 5b.$

30. $5m - 3.$

31. $6p - 5.$

32. $7x - 1.$

33. $4 - 7p.$

34. $a - (-b).$

35. $4m^2 - 5n^2.$

36. $ax - by.$

37. $2a^2x - 3b^2y.$

38. $7a^2 - 4b^2.$

39. $2a^3b - b^3.$

40. $7p^3q^2 - 8p^2q^3.$

Find the missing term in the following perfect squares :

41. $16a^2 - (\quad) + 9b^2.$

42. $81x^2 - 72xy + (\quad).$

43. $(\quad) - 48pq + 36p^2.$

44. $49m^2 - (\quad) + 121.$

45. $64p^2 - 96pq + (\quad).$

46. $(\quad) - 120xy + 144y^2.$

Find algebraically the value of :

47. $(198)^2.$

48. $(245)^2.$

49. $(396)^2.$

50. $(138)^2.$

51. $(497)^2.$

52. $(599)^2.$

Example 8. Simplify $(3m - 5n)^2 - (2m - 3n)^2.$

The given expression

$$= (9m^2 - 30mn + 25n^2) - (4m^2 - 12mn + 9n^2)$$

$$= 5m^2 - 18mn + 16n^2.$$

Simplify :

53. $(a + b)^2 - (a - b)^2.$

54. $(2a - 3b)^2 + (3a - 2b)^2.$

55. $(px + qy)^2 - (px - qy)^2.$

56. $(2mx - ny)^2 - (mx + 3ny)^2.$

57. $(3ab - 2b^2)^2 - 4(a^2 - b^2)^2.$

Example 9. Find the value of $a^2 + b^2$ when $a + b = 9$ and $ab = 20.$

$$a^2 + b^2 = (a + b)^2 - 2ab.$$

Substituting the values of $(a + b)$ and ab , we have the expression

$$= 9^2 - 2 \times 20$$

$$= 81 - 40 = 41.$$

Example 10. Find the value of $a^2 + b^2$ when $a - b = 7$ and $ab = 60$.

$$a^2 + b^2 = (a - b)^2 + 2ab.$$

Substituting the values of $(a - b)$ and ab , we have the expression

$$\begin{aligned} &= 7^2 + 2 \times 60 \\ &= 49 + 120 = 169. \end{aligned}$$

Find the value of $a^2 + b^2$ when

$$58. \quad a + b = 27 \text{ and } ab = 180. \qquad 59. \quad a + b = 19 \text{ and } ab = 88.$$

$$60. \quad a - b = 7 \text{ and } ab = 120. \qquad 61. \quad a - b = 9 \text{ and } ab = 112.$$

Formula III. $(a + b)(a - b) = a^2 - b^2$.

Proof.
$$\begin{aligned} (a + b)(a - b) &= a(a - b) + b(a - b) \\ &= a^2 - ab + ba - b^2 \\ &= a^2 - b^2. \end{aligned}$$

Statement of the formula in words :

The product of the sum and the difference of two quantities is equal to the difference of their squares.

Example 1. Multiply $4a + 3b$ by $4a - 3b$.

$$\begin{aligned} (4a + 3b)(4a - 3b) &= (4a)^2 - (3b)^2 \\ &= 16a^2 - 9b^2. \end{aligned}$$

Example 2. Multiply $a + b + c$ by $a - b - c$.

$$\begin{aligned} (a + b + c)(a - b - c) &= \{ a + (b + c) \} \{ a - (b + c) \} \\ &= a^2 - (b + c)^2 \\ &= a^2 - (b^2 + 2bc + c^2) \\ &= a^2 - b^2 - 2bc - c^2 \\ &= a^2 - b^2 - c^2 - 2bc. \end{aligned}$$

Example 3. Simplify $(x - 2y + 3z)^2 - (x + 2y - 3z)^2$.

$$\begin{aligned} \text{The given expression} &= \{ (x - 2y + 3z) + (x + 2y - 3z) \} \\ &\quad \times \{ (x - 2y + 3z) - (x + 2y - 3z) \} \\ &= 2x \times (-4y + 6z) \\ &= -8xy + 12xz. \end{aligned}$$

Example 4. Find the value of $(236)^2 - (232)^2$.

$$\begin{aligned} \text{The given expression} &= (236 + 232)(236 - 232) \\ &= 468 \times 4 = 1872. \end{aligned}$$

EXERCISE 23.

Write down the product of :

1. $3x+5$ and $3x-5$.
2. $am+bn$ and $am-bn$.
3. $ax+b^2$ and $ax-b^2$.
4. $px+qy$ and $px-qy$.
5. $4a+5b$ and $4a-5b$.
6. $3xy-7xz$ and $3xy+7xz$.
7. $a+b$, $a-b$ and a^2+b^2 .
8. $a+1$, $a-1$ and a^2+1 .
9. a^2+b^2 , a^2-b^2 and a^4+b^4 .
10. $x+y-z$ and $x-y+z$.
11. $x-3y+4z$ and $x-3y-4z$.
12. x^2-xy+y^2 and x^2+xy+y^2 .
13. x^2+x+1 and x^2-x+1 .
14. x^4+x^2+1 and x^4-x^2+1 .

Simplify :

15. $(2a-b+c)^2-(a+b-2c)^2$.
16. $(a^2+ab+b^2)^2-(a^2-ab+b^2)^2$.
17. $(a+b-c+d)^2-(a-b+c-d)^2$.
18. $(2a+3b-4c+5d)^2-(2a-3b+4c-5d)^2$.

Find the value of :

19. $(547)^2-(542)^2$.
20. $(784)^2-(779)^2$.
21. $328 \times 328 - 318 \times 318$.
22. $614 \times 614 - 607 \times 607$.

Formula IV. $(x+a)(x+b)=x^2+(a+b)x+ab$.

Proof. $(x+a)(x+b)=x(x+b)+a(x+b)$
 $=x^2+xb+ax+ab$
 $=x^2+x(a+b)+ab$.

From this formula, we can easily deduce the following results:

$$(x-a)(x-b)=x^2+(-a-b)x+(-a)(-b) \quad \text{(i)}$$

$$(x+a)(x-b)=x^2+(a-b)x+a(-b) \quad \text{(ii)}$$

$$(x-a)(x+b)=x^2+(-a+b)x+(-a)b. \quad \text{(iii)}$$

Thus the above formula can be expressed in more general terms as follows :

$$(x+a)(x+b)=x^2+(\text{algebraic sum of 2nd terms})x+\text{product of 2nd terms}.$$

It is interesting to note that formulæ I and II can also be deduced from this formula, as follows :

$$\begin{aligned}
 (a+b)^2 &= (a+b)(a+b) \\
 &= a^2 + (b+b)a + b.b \\
 &= a^2 + 2ab + b^2,
 \end{aligned}$$

and

$$\begin{aligned}
 (a-b)^2 &= (a-b)(a-b) \\
 &= a^2 + (-b-b)a + (-b)(-b) \\
 &= a^2 - 2ab + b^2.
 \end{aligned}$$

Example 1. Write down the product of:

- (i) $x+4$ and $x+5$. (ii) $x+6$ and $x-2$.
 (iii) $x-3$ and $x+7$. (iv) $x-5$ and $x-6$.

(i) Since $4+5=9$ and $4 \times 5=20$,

$$\therefore (x+4)(x+5) = x^2 + 9x + 20.$$

(ii) Since $+6-2=+4$ and $(+6)(-2)=-12$,

$$\therefore (x+6)(x-2) = x^2 + 4x - 12.$$

(iii) Since $-3+7=+4$ and $(-3)(+7)=-21$,

$$\therefore (x-3)(x+7) = x^2 + 4x - 21.$$

(iv) Since $(-5)+(-6)=-11$ and $(-5)(-6)=+30$,

$$\therefore (x-5)(x-6) = x^2 - 11x + 30.$$

EXERCISE 24.

Find out *mentally* the product of:

- | | |
|-------------------------|-------------------------|
| 1. $x+3$ and $x+5$. | 2. $x+4$ and $x+6$. |
| 3. $x+5$ and $x+7$. | 4. $x+7$ and $x+9$. |
| 5. $m-8$ and $m+5$. | 6. $m-11$ and $m+7$. |
| 7. $p-14$ and $p+9$. | 8. $a-20$ and $a+11$. |
| 9. $x-7$ and $x+15$. | 10. $x-8$ and $x+12$. |
| 11. $x-6$ and $x+17$. | 12. $x-12$ and $x+20$. |
| 13. $x+10$ and $x-11$. | 14. $x+13$ and $x-8$. |
| 15. $x+16$ and $x-9$. | 16. $x+12$ and $x-7$. |
| 17. $x+7$ and $x-17$. | 18. $x+3$ and $x-15$. |
| 19. $x-5$ and $x-18$. | 20. $x+7$ and $x-19$. |
| 21. $x-7$ and $x-9$. | 22. $x-12$ and $x-15$. |
| 23. $x-13$ and $x-10$. | 24. $x-9$ and $x-14$. |

Example 2. Multiply together $x+y+z$ and $x+y-2z$.

Putting a for $(x+y)$ in both the expressions, we have

$$\begin{aligned}(x+y+z)(x+y-2z) &= (a+z)(a-2z) \\ &= a^2 + (z-2z)a - 2z^2 \\ &= a^2 - az - 2z^2.\end{aligned}$$

Substituting the value of a , we have the product

$$\begin{aligned}&= (x+y)^2 - (x+y)z - 2z^2 \\ &= x^2 + 2xy + y^2 - xz - yz - 2z^2 \\ &= x^2 + y^2 - 2z^2 + 2xy - xz - yz.\end{aligned}$$

Multiply :

- | | |
|--------------------------------|-------------------------|
| 25. $2x+3$ and $2x+5$. | 26. $3m+5$ and $3m-2$. |
| 27. $1+2x$ and $1-3x$. | 28. $3+4x$ and $3-5x$. |
| 29. $x-y+z$ and $x-y+2z$. | |
| 30. $2x+y+3z$ and $2x+y-z$. | |
| 31. $2x+3y+z$ and $2x+3y-5z$. | |

Formula V. $(x+a)(x+b)(x+c)$

$$= x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc.$$

The formula can be proved by actual multiplication.

From this formula, we can easily deduce the following results:

$$\begin{aligned}(x+a)(x+b)(x-c) &= x^3 + (a+b-c)x^2 + \{ ab+a(-c)+b(-c) \} x + ab(-c). \quad (i) \\ (x+a)(x-b)(x-c) &= x^3 + (a-b-c)x^2 \\ &\quad + \{ a(-b)+a(-c)+(-b)(-c) \} x + a(-b)(-c). \quad (ii) \\ (x-a)(x-b)(x-c) &= x^3 + (-a-b-c)x^2 \\ &\quad + \{ (-a)(-b)+(-a)(-c)+(-b)(-c) \} x + (-a)(-b)(-c). \quad (iii)\end{aligned}$$

Statement of the formula in more general terms :

$$\begin{aligned}(x+a)(x+b)(x+c) &= x^3 + (\text{algebraic sum of 2nd terms})x^2 \\ &\quad + (\text{sum of the products of 2nd terms, taken two at a time})x \\ &\quad + \text{product of 2nd terms}.\end{aligned}$$

Example 1. Write down the product of $x+2$, $x+3$, and $x-6$.

$$\begin{aligned}(x+2)(x+3)(x-6) &= x^3 + (2+3-6)x^2 \\ &\quad + \{ 2 \times 3 + 2(-6) + 3(-6) \} x + 2 \times 3 \times (-6) \\ &= x^3 - x^2 - 24x - 36.\end{aligned}$$

Example 2. Write down the product of $x-3$, $x-4$ and $x-5$.

$$\begin{aligned}(x-3)(x-4)(x-5) &= x^3 + (-3-4-5)x^2 \\ &+ \{ (-3)(-4) + (-3)(-5) + (-4)(-5) \} x + (-3)(-4)(-5) \\ &= x^3 - 12x^2 + 47x - 60.\end{aligned}$$

EXERCISE 25.

Write down the product of :

- | | |
|----------------------------|----------------------------|
| 1. $(x+1)(x+2)(x+3)$. | 2. $(x+2)(x+3)(x+4)$. |
| 3. $(x+4)(x+5)(x+6)$. | 4. $(x+3)(x+4)(x-5)$. |
| 5. $(x+4)(x+5)(x-7)$. | 6. $(x+5)(x+2)(x-4)$. |
| 7. $(x+3)(x-4)(x-5)$. | 8. $(x+4)(x-6)(x-3)$. |
| 9. $(x-3)(x-4)(x-5)$. | 10. $(x-4)(x-1)(x-7)$. |
| 11. $(x-3)(x-5)(x-2)$. | 12. $(x-4)(x-6)(x-8)$. |
| 13. $(2x+1)(2x+3)(2x+5)$. | 14. $(3x-1)(3x-2)(3x-4)$. |
| 15. $(ax+2)(ax+3)(ax+6)$. | 16. $(mx+1)(mx-4)(mx-5)$. |
| 17. $(px-3)(px-4)(px-7)$. | |

Complete the following :

18. $(x+p)(\quad + q)(x+\quad) = x^3 + (\quad)x^2 + (\quad)x + pqr$.
19. $(\quad + a)(p+2b)(p-3c) = (p)^3 + (\quad)p^2 + (\quad)p + (\quad)$.
20. $(m+3)(\quad - 4)(m+\quad) = m^3 + (\quad)m^2 + (\quad)m - 60$.

Formula VI. $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

or

$$= a^3 + b^3 + 3ab(a+b)$$

The formula can be easily established by actual multiplication or can be deduced from formula V, as shown below :

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)(a+b) \\ &= a^3 + (b+b+b)a^2 + (b \times b + b \times b + b \times b)a + b \times b \times b \\ &= a^3 + 3ba^2 + 3b^2a + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \\ &= a^3 + b^3 + 3ab(a+b).\end{aligned}$$

Statement of the formula in words :

The cube of the sum of two quantities is equal to the sum of their cubes plus three times their product multiplied by their sum.

Cor. $a^3 + b^3 = (a+b)^3 - 3ab(a+b)$.

Example 1. Find the cube of $2x+3y$.

Reducing the expression to the standard form, we have

$$\begin{aligned}(2x+3y)^3 &= \{ (2x)+(3y) \}^3 \\ &= (2x)^3 + (3y)^3 + 3(2x \times 3y)(2x+3y) \\ &= 8x^3 + 27y^3 + 36x^2y + 54xy^2,\end{aligned}$$

or

$$\begin{aligned}(2x+3y)^3 &= (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3 \\ &= 8x^3 + 36x^2y + 54xy^2 + 27y^3.\end{aligned}$$

EXERCISE 26.

Find the cube of:

1. $2x+y$.
2. $x+3y$.
3. $2a+3b$.
4. $ax+by$.
5. $1+3x$.
6. $2+5x$.
7. $3x^2+2$.
8. a^2+b^2 .
9. $3a^2+2b^2$.

Example 2. Find the missing terms as well as the missing parts of terms in

$$(\quad + \quad)^3 = (2x)^3 + 3(2x)^2(\quad) + 3(\quad)(5y)^2 + (\quad)^3.$$

Since in the expansion the first term is $(2x)^3$ and the third term contains $(5y)^2$,

$$\therefore \text{the left-hand side} = (2x+5y)^3$$

$$\text{and the right-hand side} = (2x)^3 + 3(2x)^2(5y) + 3(2x)(5y)^2 + (5y)^3.$$

Fill in the gaps in the following:

$$10. (\quad + \quad)^3 = (4m)^3 + 3(\quad)^2(\quad) + 3(\quad)(\quad)^2 + (2n)^3.$$

$$11. (2p + \quad)^3 = (\quad)^3 + 3(\quad)(\quad) + 3(2p)(\quad) + (3q)^3.$$

$$12. (\quad + \quad)^3 = (5p)^3 + 3(\quad)(5p)^2 + 3(\quad)(2q)^2 + (\quad)^3.$$

$$13. (2a + \quad)^3 = (\quad)^3 + (3b)^3 + 3(\quad)(2a+3b).$$

Example 3. Simplify:

$$(x+2y)^3 + (2x-y)^3 + 3(3x+y)(x+2y)(2x-y).$$

$$\begin{aligned}\text{Putting } a \text{ for } x+2y \text{ and } b \text{ for } 2x-y, \text{ we have the expres-} \\ \text{sion} &= a^3 + b^3 + 3(a+b)(a)(b), \quad [\because a+b=3x+y.] \\ &= (a+b)^3.\end{aligned}$$

Substituting the values of a and b , we have the expression

$$\begin{aligned}&= \{ (x+2y) + (2x-y) \}^3 \\ &= (3x+y)^3\end{aligned}$$

$$= (3x)^3 + 3(3x)^2y + 3(3x)y^2 + y^3$$

$$= 27x^3 + 27x^2y + 9xy^2 + y^3.$$

Example 4. If $a + b = 6$ and $ab = 8$, find the value of $a^3 + b^3$.
 $a^3 + b^3 = (a + b)^3 - 3ab(a + b).$

Substituting the values of $(a + b)$ and ab , we have the expression

$$= 6^3 - 3 \times 8 \times 6$$

$$= 216 - 144 = 72.$$

Simplify :

14. $(3a + 5b)^3 + (4a - 3b)^3 + 3(3a + 5b)(4a - 3b)(7a + 2b).$

15. $(2x - 7y)^3 + (3x + 12y)^3 + 15(2x - 7y)(3x + 12y)(x + y).$

16. $(a + 2)^3 + (a + 1)^3.$ 17. $(2a + 3)^3 - (2a + 1)^3.$

Find the value of $a^3 + b^3$ when

18. $a + b = 5$ and $ab = 6.$ 19. $a + b = 7$ and $ab = 12.$

20. $a + b = 9$ and $ab = 18.$

Example 5. Find the value of $64 + 144x + 108x^2 + 27x^3$ when $x = -3$.

$$64 + 144x + 108x^2 + 27x^3 = (4)^3 + 3(4)^2(3x) + 3(4)(3x)^2 + (3x)^3$$

$$= (4 + 3x)^3 = (4 - 9)^3 = (-5)^3 = -125.$$

Find the value of:

21. $x^3 + 6x^2 + 12x + 20$ when $x = 5.$

22. $8x^3 + 36x^2 + 54x + 35$ when $x = -3.$

23. $27x^3 + 27x^2 + 9x + 21$ when $x = -2.$

24. If $a + b = 7$, shew that $a^3 + b^3 + 21ab = 343.$

25. If $a + b = 8$, shew that $a^3 + b^3 + 24ab = 512.$

Formula VII. $(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$
or $= a^3 - b^3 - 3ab(a - b).$

The formula can be proved by actual multiplication or by putting $(-b)$ for $(+b)$ in formula VI, as shown below:

$$(a - b)^3 = \{ a + (-b) \}^3 = a^3 + 3a^2(-b) + 3a(-b)^2 + (-b)^3$$

$$= a^3 - 3a^2b + 3ab^2 - b^3$$

$$= a^3 - b^3 - 3ab(a - b).$$

Statement of the formula in words :

The cube of the difference of two quantities is equal to the difference of their cubes minus three times their product multiplied by their difference.

Cor. $a^3 - b^3 = (a - b)^3 + 3ab(a - b).$

Example 6. Find the cube of $3x - 2y$.

$$\begin{aligned}(3x - 2y)^3 &= (3x)^3 - (2y)^3 - 3(3x)(2y)(3x - 2y) \\ &= 27x^3 - 8y^3 - 54x^2y + 36xy^2 \\ &= 27x^3 - 54x^2y + 36xy^2 - 8y^3.\end{aligned}$$

Example 7. If $a - b = 4$ and $ab = 12$, find the value of $a^3 - b^3$.

$$\begin{aligned}a^3 - b^3 &= (a - b)^3 + 3ab(a - b) \\ &= 4^3 + 3 \times 12 \times 4 \\ &= 64 + 144 = 208.\end{aligned}$$

Example 8. Find the value of $8 - 9x + 27x^2 - 27x^3$ when $x = 4$.

$$\begin{aligned}\text{The given expression} &= 1 - 3(3x) + 3(3x)^2 - (3x)^3 + 7 \\ &= (1 - 3x)^3 + 7 = (1 - 12)^3 + 7 \\ &= (-11)^3 + 7 = -1331 + 7 \\ &= -1324.\end{aligned}$$

Find the cube of :—

26. $2x - 1$. 27. $3 - 2x$. 28. $3a - 4b$. 29. $4x - 3y$,
30. $5x - 4y$. 31. $4m^2 - 3n$. 32. $2ax - 3by$.

Fill in the gaps in the following :

33. $(\quad - 3y)^3 = (2x)^3 - 3(\quad)^2(\quad) + 3(\quad)(\quad)^2 - (3y)^3$.
34. $(4m - \quad)^3 = (\quad)^3 - 3(\quad)^2(\quad) + 3(4m)(\quad)^2 - (5x)^3$.
35. $(\quad - 6b)^3 = (\quad)^3 - 3(4a)^2(\quad) + 3(\quad)(6b)^2 - (\quad)^3$.
36. $(2p - \quad)^3 = (\quad)^3 - (\quad)^3 - 3(\quad)(\quad)(2p - 3q)$.

Simplify :

37. $(6m - 7n)^3 - (3m + 5n)^3 - 9(6m - 7n)(3m + 5n)(m - 4n)$.
38. $(5p + 3q)^3 - (4p - 2q)^3 - 3(5p + 3q)(4p - 2q)(p + 5q)$.
39. $(x + 3)^3 - (x - 1)^3$. 40. $(2x + 3)^3 - (2x - 5)^3$.

Find the value of $a^3 - b^3$ when

41. $a - b = 3$ and $ab = 18$. 42. $a - b = 5$ and $ab = 14$.
43. $a - b = 4$ and $ab = 32$.

Find the value of :

44. $p^3 - 6p^2 + 12p - 12$ when $p = 3$.

45. $8p^3 - 36p^2 + 54p - 35$ when $p = 2$.

46. $27p^3 - 54p^2 + 36p - 20$ when $p = 4$.

Formula VIII. $(a + b)(a^2 - ab + b^2) = a^3 + b^3$.

Proof. $(a + b)(a^2 - ab + b^2) = a(a^2 - ab + b^2) + b(a^2 - ab + b^2)$
 $= a^3 - a^2b + ab^2 + ba^2 - ab^2 + b^3$
 $= a^3 + b^3$.

Conversely, $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

Example 1. Find the product of $3a + 2b$ and $9a^2 - 6ab + 4b^2$.

$$\begin{aligned} (3a + 2b)(9a^2 - 6ab + 4b^2) &= \{ (3a) + (2b) \} \{ (3a)^2 - (3a)(2b) \\ &\quad + (2b)^2 \} \\ &= (3a)^3 + (2b)^3 \\ &= 27a^3 + 8b^3. \end{aligned}$$

Example 1. Simplify

$$(2x + 3)(4x^2 - 6x + 9) - (3x + 2)(9x^2 - 6x + 4).$$

Since $(2x + 3)(4x^2 - 6x + 9) = \{ (2x) + (3) \} \{ (2x)^2 - (2x)(3) + (3)^2 \}$
 $= (2x)^3 + (3)^3 = 8x^3 + 27,$

and $(3x + 2)(9x^2 - 6x + 4) = \{ (3x) + (2) \} \{ (3x)^2 - (3x)(2) + (2)^2 \}$
 $= (3x)^3 + (2)^3$
 $= 27x^3 + 8,$

\therefore the given expression $= (8x^3 + 27) - (27x^3 + 8)$
 $= 8x^3 + 27 - 27x^3 - 8$
 $= 19 - 19x^3.$

EXERCISE 27.

Find the product of :

1. $x + 3$ and $x^2 - 3x + 9$.
2. $x + 5$ and $x^2 - 5x + 25$.
3. $x + 7$ and $x^2 - 7x + 49$.
4. $x + 11$ and $x^2 - 11x + 121$.
5. $25a^2 - 15a + 9$ and $5a + 3$.
6. $9m^2 - 15mn + 25n^2$ and $3m + 5n$.
7. $x^2y^2 - 4xyz + 16z^2$ and $xy + 4z$.
8. $4x + 9y$ and $16x^2 - 36xy + 81y^2$.
9. $a^2 + bc$ and $a^3 - a^2bc + b^2c^2$.

10. $a^4 - 4a^2b^2 + 16b^4$ and $a^2 + 4b^2$.

11. $x^2 + 1$ and $x^4 - x^2 + 1$.

Simplify :

12. $(x+5)(x^2-5x+25) - (x+3)(x^2-3x+9)$.

13. $(x+2y)(x^2-2xy+4y^2) + (2x+y)(4x^2-2xy+y^2)$.

14. $(4x+3y)(16x^2-12xy+9y^2) - (3x+4y)(9x^2-12xy+16y^2)$.

15. $(x^2+4)(x^4-4x^2+16) - (x^2+2)(x^4-2x^2+4)$.

Formula IX. $(a-b)(a^2+ab+b^2) = a^3 - b^3$.

Proof. $(a-b)(a^2+ab+b^2) = a(a^2+ab+b^2) - b(a^2+ab+b^2)$
 $= a^3 + a^2b + ab^2 - ba^2 - ab^2 - b^3$
 $= a^3 - b^3$.

Conversely, $a^3 - b^3 = (a-b)(a^2+ab+b^2)$.

Example 3. Find the product of $3xy - z$ and $9x^2y^2 - 3xyz + z^2$.

$$(3xy - z)(9x^2y^2 + 3xyz + z^2) = \{ (3xy) - z \} \{ (3xy)^2 + (3xy)z + z^2 \}$$

$$= (3xy)^3 - z^3 = 27x^3y^3 - z^3.$$

Example 4. Simplify $(x-3)(x^2+3x+9) - (x-2)(x^2+2x+4)$.

$$(x-3)(x^2+3x+9) = \{ (x) - (3) \} \{ (x)^2 + (x)(3) + (3)^2 \}$$

$$= x^3 - 3^3 = x^3 - 27,$$

$$(x-2)(x^2+2x+4) = \{ (x) - (2) \} \{ (x)^2 + (x)(2) + (2)^2 \}$$

$$= x^3 - 2^3 = x^3 - 8.$$

$$\therefore \text{the given expression} = (x^3 - 27) - (x^3 - 8)$$

$$= x^3 - 27 - x^3 + 8.$$

$$= -19.$$

Write down the product of :

16. $(x-2)(x^2+2x+4)$. 17. $(x-6)(x^2+6x+36)$.

18. $(x-8)(x^2+8x+64)$. 19. $(x-12)(x^2+12x+144)$.

20. $(3x-2)(9x^2+6x+4)$. 21. $(5x-3y)(25x^2+15xy+9y^2)$.

22. $(pq-2r)(p^2q^2+2pqr+4r^2)$.

23. $(a^2-bc)(a^4+a^2bc+b^2c^2)$.

Simplify :

24. $(3m-4n)(9m^2+12mn+16n^2) - (2m-3n)(4m^2+6mn+9n^2)$.

25. $(4p-3q)(16p^2+12pq+9q^2) - (3p-4q)(9p^2+12pq+16q^2)$.

26. $(m^2-6)(m^4+6m^2+36) - (m^2-3)(m^4+3m^2+9)$.

Fill in the gaps :

$$27. (a)^3 + ()^3 = \{ () + (3b) \} \{ (a)^2 - () () + ()^2 \}.$$

$$28. ()^3 - (5y)^3 = \{ (2x) - () \} \{ ()^2 + () () + (5y)^2 \}.$$

$$29. (4p)^3 + ()^3 = \{ () + () \} \{ ()^2 - (4p)(3q) + ()^2 \}.$$

$$30. ()^3 - (6b)^3 = \{ () - () \} \{ 49a^2 + () () + 36b^2 \}.$$

Formula X. Square of multinomial.

$$\begin{aligned} (a+b+c)^2 &= \{ (a+b) + c \}^2 \\ &= (a+b)^2 + 2(a+b)c + c^2 \\ &= a^2 + 2ab + b^2 + 2ac + 2bc + c^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc. \end{aligned}$$

Similarly, $(a+b+c+d)^2$

$$= \{ (a+b) + (c+d) \}^2$$

$$= (a+b)^2 + 2(a+b)(c+d) + (c+d)^2$$

$$= a^2 + b^2 + 2ab + 2ac + 2ad + 2bc + 2bd + c^2 + d^2 + 2cd$$

$$= a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$

From such examples, we establish the following rule :

The square of any multinomial is equal to the sum of the squares of its terms plus twice the sum of the products of the terms, taken two at a time.

Example 1. Find the square of $2a - 3b + c$.

$$\begin{aligned} (2a - 3b + c)^2 &= (2a)^2 + (-3b)^2 + c^2 + 2(2a)(-3b) + 2(2a)(c) \\ &\quad + 2(-3b)(c) \\ &= 4a^2 + 9b^2 + c^2 - 12ab + 4ac - 6bc. \end{aligned}$$

Example 2. Find the square of $a - 3b + c - 2d$.

$$\begin{aligned} (a - 3b + c - 2d)^2 &= a^2 + (-3b)^2 + c^2 + (-2d)^2 + 2a(-3b) \\ &\quad + 2ac + 2a(-2d) + 2(-3b)c + 2(-3b)(-2d) \\ &\quad + 2c(-2d) \\ &= a^2 + 9b^2 + c^2 + 4d^2 - 6ab + 2ac - 4ad - 6bc \\ &\quad + 12bd - 4cd. \end{aligned}$$

EXERCISE 28.

Write down the square of :

1. $a + b - c$.

2. $a - b - c$.

3. $2a - b - c$.

4. $2a + b - c$. 5. $2a - b + 3c$. 6. $5x - 2y + 3z$.
 7. $2x - 3y + 1$. 8. $x^2 - xy + y^2$. 9. $px^2 - qxy + ry^2$.
 10. $2a + b + 3c + d$. 11. $a - 2b - 3c + 4d$.
 12. $2p - 3q + r - 2s + t$. 13. $-3x - 4y + 2z - v$.

Simplify :

14. $(2a - b + c)^2 - (a + b - 2c)^2$.
 15. $(2a + b + c)^2 - (a - b - c)^2 - (a + b - c)^2$.
 16. $(a + 2b + c)^2 - (a - b + 2c)^2 + (2a + b - c)^2$.
 17. $(a + b + c + d)^2 + (a - b + c - d)^2$.
 18. $(a - b - c - d)^2 + (a + b - c - d)^2$.
 19. $(2a - b + c - d)^2 - (a + b - c + d)^2 - (a - b + c - d)^2$.

Example 3. Find the value of $a^2 + 9b^2 - 6ab - 2a + 6b + 64$ when $a = 33$ and $b = 24$.

$$\begin{aligned} \text{The given expression} &= (a)^2 + (-3b)^2 + 2(a)(-3b) + 2(a)(-1) \\ &\quad + 2(-3b)(-1) + (-1)^2 + 63 \\ &= (a - 3b - 1)^2 + 63 \end{aligned}$$

Substituting the values of a and b , we have the expression

$$\begin{aligned} &= (33 - 72 - 1)^2 + 63 \\ &= (-40)^2 + 63 \\ &= 1600 + 63 = 1663. \end{aligned}$$

Find the value of :

20. $x^2 + y^2 + z^2 - 2xy + 2xz - 2yz$

when $x = 15$, $y = 20$, $z = 24$.

21. $p^2 + 4q^2 + 9z^2 - 4pq + 6pz - 12qz$

when $p = 8$, $q = 12$, $z = 15$.

22. $x^2 + 4y^2 - 4xy + 10x - 20y + 36$ when $x = 16$, $y = 9$.

23. $9a^2 + b^2 - 6ab + 6a - 2b - 28$ when $a = 15$, $b = 37$.

A few important results, which can be deduced from formulæ I and II, are given below. They may be learnt by heart.

Formula XI. $(a + b)^2 = (a - b)^2 + 4ab$.

Proof. $(a + b)^2 = a^2 + b^2 + 2ab$

$$= (a^2 + b^2 - 2ab) + 4ab$$

$$= (a - b)^2 + 4ab.$$

Formula XII. $(a-b)^2 = (a+b)^2 - 4ab$.

$$\begin{aligned} \text{Proof. } (a-b)^2 &= a^2 + b^2 - 2ab \\ &= (a^2 + b^2 + 2ab) - 4ab \\ &= (a+b)^2 - 4ab. \end{aligned}$$

Formula XIII. $(a+b)^2 - (a-b)^2 = 4ab$.

$$\begin{aligned} \text{Proof. } (a+b)^2 - (a-b)^2 &= (a^2 + 2ab + b^2) - (a^2 - 2ab + b^2) \\ &= 4ab. \end{aligned}$$

Formula XIV. $\frac{1}{2} \{ (a-b)^2 + (b-c)^2 + (c-a)^2 \}$
 $= a^2 + b^2 + c^2 - ab - ac - bc$.

$$\begin{aligned} \text{Proof. } \frac{1}{2} \{ (a-b)^2 + (b-c)^2 + (c-a)^2 \} \\ &= \frac{1}{2} \{ (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ca + a^2) \} \\ &= \frac{1}{2} \{ 2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca \} \\ &= a^2 + b^2 + c^2 - ab - ac - bc. \end{aligned}$$

Example 1. Find the value of $(a+b)^2$ when $a-b=3$ and $ab=10$.

$$\begin{aligned} (a+b)^2 &= (a-b)^2 + 4ab \\ &= 3^2 + 4 \times 10 = 9 + 40 = 49. \end{aligned}$$

Example 2. Find the value of ab when $a+b=9$ and $a-b=3$.

Since

\therefore

$$\begin{aligned} 4ab &= (a+b)^2 - (a-b)^2, \\ ab &= \frac{1}{4} \{ (a+b)^2 - (a-b)^2 \} \\ &= \frac{1}{4} (9^2 - 3^2) \\ &= \frac{1}{4} (81 - 9) = 18. \end{aligned}$$

Example 3. Find the value of $a^2 + b^2 + c^2 - ab - ac - bc$ when $a=32$, $b=30$ and $c=27$.

$$\begin{aligned} a^2 + b^2 + c^2 - ab - ac - bc &= \frac{1}{2} \{ (a-b)^2 + (b-c)^2 + (c-a)^2 \} \\ &= \frac{1}{2} \{ 2^2 + 3^2 + (-5)^2 \} = \frac{1}{2} (4 + 9 + 25) \\ &= 19. \end{aligned}$$

EXERCISE 29.

Find the value of :

1. $(a+b)^2$ when $a-b=5$, $ab=36$.
2. $(a+b)^2$ when $a-b=4$, $ab=77$.
3. $(a-b)^2$ when $a+b=12$, $ab=35$.
4. $(a-b)^2$ when $a+b=15$, $ab=36$.
5. ab when $a+b=11$, $a-b=3$.

6. ab when $a+b=22$, $a-b=8$.
 7. $a^2+b^2+c^2-ab-ac-bc$ when $a=15$, $b=16$, $c=17$.
 8. $a^2+b^2+c^2-ab-ac-bc$ when $a=1.8$, $b=1.3$, $c=.9$.
 9. $a^2+b^2+c^2-ab-ac-bc$ when $a=p+1$, $b=p+2$,
 $c=p+3$.
 10. Simplify $(x+1)^2+(x+2)^2+(x+3)^2-(x+1)(x+2)$
 $-(x+1)(x+3)-(x+2)(x+3)$.
 [Hint. Put a for $x+1$, b for $x+2$ and c for $x+3$.]
 11. Simplify $(x+3)^2+(x+5)^2+(x+7)^2-(x+3)(x+5)$
 $-(x+3)(x+7)-(x+5)(x+7)$.

Example 4. If $x + \frac{1}{x} = 6$, find the value of :

(i) $x^2 + \frac{1}{x^2}$, (ii) $x^4 + \frac{1}{x^4}$.

$$\begin{aligned} \text{(i)} \quad \left(x^2 + \frac{1}{x^2}\right) &= \left(x^2 + \frac{1}{x^2} + 2\right) - 2 \\ &= \left(x + \frac{1}{x}\right)^2 - 2 \\ &= 6^2 - 2 = 34. \end{aligned}$$

$$\begin{aligned} \text{(ii)} \quad \left(x^4 + \frac{1}{x^4}\right) &= \left(x^4 + \frac{1}{x^4} + 2\right) - 2 \\ &= \left(x^2 + \frac{1}{x^2}\right)^2 - 2 \\ &= 34^2 - 2 = 1154. \end{aligned}$$

Example 5. If $x - \frac{1}{x} = 4$, find the value of $x^3 - \frac{1}{x^3}$.

$$\begin{aligned} x^3 - \frac{1}{x^3} &= (x)^3 - \left(\frac{1}{x}\right)^3 \\ &= \left(x - \frac{1}{x}\right)^3 + 3(x)\left(\frac{1}{x}\right)\left(x - \frac{1}{x}\right) \\ &= \left(x - \frac{1}{x}\right)^3 + 3\left(x - \frac{1}{x}\right) \\ &= 4^3 + 3 \times 4 \\ &= 64 + 12 = 76. \end{aligned}$$

Find the value of :

*12. $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$ when $x + \frac{1}{x} = 5$.

*13. $x^2 + \frac{1}{x^2}$ and $x^4 + \frac{1}{x^4}$ when $x - \frac{1}{x} = 3$.

*14. $x^4 + \frac{1}{x^4}$ when $x - \frac{1}{x} = p$.

*15. $x^3 + \frac{1}{x^3}$ when $x + \frac{1}{x} = 5$.

*16. $x^3 - \frac{1}{x^3}$ when $x - \frac{1}{x} = 2$.

*17. $x^3 + \frac{1}{x^3}$ when $x + \frac{1}{x} = m$.

*18. $x^3 - \frac{1}{x^3}$ when $x - \frac{1}{x} = a$.

Example 6. Express $(x + 5y + z)^2 - 4(x + 3y)(2y + z)$ as a perfect square.

Let $x + 3y = a$ and $2y + z = b$, then $x + 5y + z = a + b$.

$$\begin{aligned} \therefore \text{the given expression} &= (a + b)^2 - 4ab = (a - b)^2 \\ &= \{ (x + 3y) - (2y + z) \}^2 \\ &= (x + y - z)^2. \end{aligned}$$

Example 7. If $x = a + m$, $y = b + m$ and $z = c + m$, prove that

$$\begin{aligned} x^2 + y^2 + z^2 - xy - xz - yz &= a^2 + b^2 + c^2 - ab - ac - bc. \\ x^2 + y^2 + z^2 - xy - xz - yz &= \frac{1}{2} \{ (x - y)^2 + (y - z)^2 + (z - x)^2 \} \\ &= \frac{1}{2} \{ (a + m - b - m)^2 + (b + m - c - m)^2 \\ &\quad + (c + m - a - m)^2 \} \\ &= \frac{1}{2} \{ (a - b)^2 + (b - c)^2 + (c - a)^2 \} \\ &= a^2 + b^2 + c^2 - ab - ac - bc. \end{aligned}$$

Express as a perfect square :

*19. $(3x - y + z)^2 + 4(3x + y)(2y - z)$.

*20. $(2x + y - z)^2 + 4(2x + 3y)(2y + z)$.

*21. $(x + 3y + 3z)^2 - 4(x + y + 2z)(2y + z)$,

*22. $(x - z)^2 - 4(x - y + z)(y - 2z)$.

*23. If $a = p + q$, $b = q + r$ and $c = r + p$, prove that
 $a^2 + b^2 + c^2 - ab - ac - bc = p^2 + q^2 + r^2 - pq - pr - qr$.

- *24. If $x = a + b - c$, $y = b + c - a$ and $z = c + a - b$, prove that
 $x^2 + y^2 + z^2 - xy - xz - yz = 4(a^2 + b^2 + c^2 - ab - ac - bc)$.
- *25. Find the value of $x^2 + y^2 + z^2 - xy - xz - yz$ when
 $x = a(a + 7)$, $y = (a + 2)(a + 5)$, $z = (a + 3)(a + 4)$.
- *26. Find the value of $8xy(x^2 + y^2)$ when $x + y = 15$ and
 $x - y = 11$.

Formula XV. $(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$
 $= a^3 + b^3 + c^3 - 3abc.$

The formula can be easily established by actual multiplication; another form of the formula, which is sometimes useful, is
 $\frac{1}{2}(a + b + c) \{ (a - b)^2 + (b - c)^2 + (c - a)^2 \} = a^3 + b^3 + c^3 - 3abc$,
 for $a^2 + b^2 + c^2 - ab - ac - bc = \frac{1}{2} \{ (a - b)^2 + (b - c)^2 + (c - a)^2 \}$
 by formula XIV.

Example 1. Simplify

$$(2x + 3y - z)(4x^2 + 9y^2 + z^2 - 6xy + 2xz + 3yz).$$

Reducing the given expression to the standard form, we have

$$\begin{aligned} & \{ (2x) + (3y) + (-z) \} \{ (2x)^2 + (3y)^2 + (-z)^2 - (2x)(3y) - (2x)(-z) \\ & \quad - (3y)(-z) \} \\ & = (2x)^3 + (3y)^3 + (-z)^3 - 3(2x)(3y)(-z) \\ & = 8x^3 + 27y^3 - z^3 + 18xyz. \end{aligned}$$

Example 2. Find the value of $a^3 + b^3 + c^3 - 3abc$ when

$$a + b + c = 9 \text{ and } ab + ac + bc = 26.$$

$$\text{Since } a + b + c = 9, \therefore (a + b + c)^2 = 81$$

$$\therefore (a^2 + b^2 + c^2) + 2(ab + ac + bc) = 81.$$

Substituting the value of $ab + ac + bc$, we have

$$(a^2 + b^2 + c^2) + 2 \times 26 = 81,$$

$$\therefore a^2 + b^2 + c^2 = 29.$$

$$\begin{aligned} \text{Thus } a^3 + b^3 + c^3 - 3abc &= (a + b + c) \{ (a^2 + b^2 + c^2) \\ & \quad - (ab + ac + bc) \} \\ &= 9(29 - 26) \\ &= 9 \times 3 = 27. \end{aligned}$$

EXERCISE 30.

Simplify :

1. $(x+y-1)(x^2+y^2+1-xy+x+y)$.
2. $(x-y-1)(x^2+y^2+1+xy+x-y)$.
3. $(2x+y+1)(4x^2+y^2+1-2xy-2x-y)$.
4. $(x+2y-z)(x^2+4y^2+z^2-2xy+xz+2yz)$.
5. $(x+2y-3z)(x^2+4y^2+9z^2-2xy+3xz+6yz)$.
6. $(2x-y-3z)(4x^2+y^2+9z^2+2xy+6xz-3yz)$.

Complete the following identities :

7. $(3x-y-z)(\quad) = 27x^3 - y^3 - z^3 - 9xyz$.
8. $(\quad)(25x^2 + 9y^2 + z^2 - 15xy + 5xz + 3yz)$
 $= 125x^3 + 27y^3 - z^3 + 45xyz$.

Find the value of $a^3 + b^3 + c^3 - 3abc$ when

9. $a+b+c=13$, $ab+ac+bc=71$.
10. $a+b+c=21$, $a^2+b^2+c^2=149$.
11. $ab+bc+ca=121$, $a^2+b^2+c^2=158$.
12. If $x=a^2-bc$, $y=b^2-ca$, $z=c^2-ab$, shew that

$$ax+by+cz=(a+b+c)(x+y+z).$$

***Example 3.** If $x=a+b$, $y=b+c$, $z=c+a$, prove that

$$x^3+y^3+z^3-3xyz=2(a^3+b^3+c^3-3abc).$$

$$x^3+y^3+z^3-3xyz=\frac{1}{2}(x+y+z) \{ (x-y)^2 + (y-z)^2 + (z-x)^2 \}.$$

Substituting the values of x, y, z , we get the expression

$$= \frac{1}{2} \{ 2(a+b+c) \} \{ (a-c)^2 + (b-a)^2 + (c-b)^2 \}$$

$$= 2 \left[\frac{1}{2}(a+b+c) \{ (a-b)^2 + (b-c)^2 + (c-a)^2 \} \right]$$

$$= 2(a^3+b^3+c^3-3abc).$$

- *13. If $a=y+z-x$, $b=z+x-y$ and $c=x+y-z$, prove that $a^3+b^3+c^3-3abc=4(x^3+y^3+z^3-3xyz)$.
- *14. If $a=y+z-3x$, $b=z+x-3y$ and $c=x+y-3z$, prove that $a^3+b^3+c^3-3abc=-16(x^3+y^3+z^3-3xyz)$.
- *15. If $2s=a+b+c$, prove that $(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c)=\frac{1}{2}(a^3+b^3+c^3-3abc)$.

2. We add here a few important results, known as **division formulae**, which should be carefully learnt by heart.

(i) $a^n - b^n$ is exactly divisible by $a - b$, if n is *any* integer.

$$\begin{aligned} \text{Thus, } \frac{a-b}{a-b} &= 1, & \frac{a^2-b^2}{a-b} &= a+b, \\ \frac{a^3-b^3}{a-b} &= a^2+ab+b^2, & \frac{a^4-b^4}{a-b} &= a^3+a^2b+ab^2+b^3, \\ \frac{a^5-b^5}{a-b} &= a^4+a^3b+a^2b^2+ab^3+b^4, \text{ and so on.} \end{aligned}$$

(ii) $a^n - b^n$ is exactly divisible by $a + b$, if n is an *even* integer.

$$\begin{aligned} \text{Thus, } \frac{a^2-b^2}{a+b} &= a-b, & \frac{a^4-b^4}{a+b} &= a^3-a^2b+ab^2-b^3, \\ \frac{a^6-b^6}{a+b} &= a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5, \text{ and so on.} \end{aligned}$$

(iii) $a^n + b^n$ is exactly divisible by $a + b$, if n is an *odd* integer.

$$\begin{aligned} \text{Thus, } \frac{a+b}{a+b} &= 1, & \frac{a^3+b^3}{a+b} &= a^2-ab+b^2, \\ \frac{a^5+b^5}{a+b} &= a^4-a^3b+a^2b^2-ab^3+b^4, \text{ and so on.} \end{aligned}$$

(iv) $a^n + b^n$ is never exactly divisible by $a + b$ or $a - b$, if n is an *even* integer.

Example 1. Simplify $\frac{8x^3+27y^3}{2x+3y}$.

$$\begin{aligned} \frac{8x^3+27y^3}{2x+3y} &= \frac{(2x)^3+(3y)^3}{2x+3y} = (2x)^2 - (2x)(3y) + (3y)^2 \\ &= 4x^2 - 6xy + 9y^2. \end{aligned}$$

Example 2. Simplify $\frac{x^5+32y^5}{x+2y}$.

$$\begin{aligned} \frac{x^5+32y^5}{x+2y} &= \frac{(x)^5+(2y)^5}{x+2y} \\ &= x^4 - x^3(2y) + x^2(2y)^2 - x(2y)^3 + (2y)^4 \\ &= x^4 - 2x^3y + 4x^2y^2 - 8xy^3 + 16y^4. \end{aligned}$$

Example 3. Simplify $\frac{a^3 + (b+c)^3}{a+b+c}$.

$$\begin{aligned}\frac{a^3 + (b+c)^3}{a+b+c} &= \frac{(a)^3 + (b+c)^3}{a+(b+c)} = a^2 - a(b+c) + (b+c)^2 \\ &= a^2 - ab - ac + b^2 + 2bc + c^2 \\ &= a^2 + b^2 + c^2 - ab - ac + 2bc.\end{aligned}$$

EXERCISE 31.

Write down by *inspection* the quotient of :

1. $(8a^3 - x^3) \div (2a - x)$.
2. $(a^3 + 64b^3) \div (a + 4b)$.
3. $(64a^6 - x^6) \div (2a + x)$.
4. $(27m^3 - 1) \div (3m - 1)$.
5. $(x^4 - 256y^4) \div (x - 4y)$.
6. $(1 - 16m^4) \div (1 + 2m)$.
7. $(32a^5 + b^5) \div (2a + b)$.
8. $(m^{18} - n^{12}) \div (m^3 + n^2)$.
9. $\left(\frac{1}{27}a^3 + b^3\right) \div \left(\frac{1}{3}a + b\right)$.
10. $(x^8 - y^4z^4) \div (x^2 + yz)$.
11. $\{ p^2 - (q - r)^2 \} \div (p - q + r)$.
12. $\{ a^3 - (b - c)^3 \} \div (a - b + c)$.
13. $(m^{18} - n^{12}) \div (m^3 - n^2)$.
14. $(x^9 + a^9) \div (x + a)$.
15. Write down by *inspection*, the continued product of $(x + a)$, $(x^2 + a^2)$, $(x^4 + a^4)$.

[*Hint.* Include $x - a$ in the continued product and divide the product so obtained by $x - a$.]

A systematised list of multiplication formulae :

1. $(a+b)^2 = a^2 + 2ab + b^2.$
2. $(a-b)^2 = a^2 - 2ab + b^2.$
3. $a^2 + b^2 = (a+b)^2 - 2ab.$
4. $a^2 + b^2 = (a-b)^2 + 2ab.$
5. $(a+b)^2 = (a-b)^2 + 4ab.$
6. $(a-b)^2 = (a+b)^2 - 4ab.$
7. $(a+b)^2 - (a-b)^2 = 4ab.$
8. $(a+b)(a-b) = a^2 - b^2.$
9. $(a+b)(a^2 - ab + b^2) = a^3 + b^3.$
10. $(a-b)(a^2 + ab + b^2) = a^3 - b^3.$
11. $(x+a)(x+b) = x^2 + (a+b)x + ab.$
12. $(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc.$
13. $(x+a)^3 = x^3 + 3x^2a + 3xa^2 + a^3, \text{ or}$
 $= x^3 + a^3 + 3xa(x+a).$
14. $(x-a)^3 = x^3 - 3x^2a + 3xa^2 - a^3, \text{ or}$
 $= x^3 - a^3 - 3xa(x-a).$
15. $x^3 + a^3 = (x+a)^3 - 3xa(x+a).$
16. $x^3 - a^3 = (x-a)^3 + 3xa(x-a).$
17. $(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.$
18. $(a+b+c+d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad$
 $+ 2bc + 2bd + 2cd.$
19. $\frac{1}{2} \{ (a-b)^2 + (b-c)^2 + (c-a)^2 \} = a^2 + b^2 + c^2 - ab - ac - bc.$
20. $(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc) = a^3 + b^3 + c^3 - 3abc,$
 $\text{or } \frac{1}{2}(a+b+c) \{ (a-b)^2 + (b-c)^2 + (c-a)^2 \} = a^3 + b^3 + c^3 - 3abc.$

CHAPTER V

FACTORS

1. **Type I.** Factors of expressions of the form $ab + ac$.

Example 1. Resolve into factors $3a^2 - 6a$.

Here $3a$ is common to both the terms : hence if we divide the given expression by $3a$, the quotient will be another factor. Thus:

$$\begin{array}{r} 3a \overline{) 3a^2 - 6a} \\ \underline{3a^2} \\ a - 2 \end{array}$$

$$\therefore 3a^2 - 6a = 3a(a - 2).$$

Example 2. Resolve into factors $2a^3b^2c - 4ab^3c^2 + 6a^2bc^3$.

Here we notice that $2abc$ is a factor in every term,

$$\begin{aligned} \therefore 2a^3b^2c - 4ab^3c^2 + 6a^2bc^3 \\ = 2abc(a^2b - 2b^2c + 3ac^2). \end{aligned}$$

[Multiply $a^2b - 2b^2c + 3ac^2$ by $2abc$ and see if the product is $2a^3b^2c - 4ab^3c^2 + 6a^2bc^3$.]

EXERCISE 32. [Mainly oral]

Resolve into factors :

- | | |
|-------------------------------------|------------------------------------|
| 1. $ma + mb$, | 2. $pr + ps$. |
| 3. $x^2 + 2x$. | 4. $3x^2 - 6$. |
| 5. $m^2n + mn^2$. | 6. $abc - bcd$. |
| 7. $-8x + 24$. | 8. $-11a^2 - 44a$. |
| 9. $75.11 + 75.3$. | 10. $49.15 - 49.12$. |
| 11. $pa + qa - ra$. | 12. $-2x^2 + 4x - 6$. |
| 13. $ax^3 + bx^2 + cx$. | 14. $-3p^3 - 6p^2 + 9p$. |
| 15. $a^2x^3 - 2a^3x^2 + a^2x^2$. | 16. $2mp^2 + 4mp - 6m$. |
| 17. $4a^5b - 6a^4b + 8a^3b^3$. | 18. $-36x^{10}y^8 + 72x^8y^{10}$. |
| 19. $ax^3y^2 + a^2xy^3 + a^3x^2y$. | 20. $33a^3b^5 - 55a^5b^3$. |

Instead of a simple expression, we may have a compound expression as a factor of every term, as in the next example.

Example 3. Resolve into factors

$$a^2(x-y) + 3ab(x-y) + 5b^2(x-y).$$

Here $x-y$ is a factor of every term,

$$\begin{aligned} \therefore a^2(x-y) + 3ab(x-y) + 5b^2(x-y) \\ = (x-y)(a^2 + 3ab + 5b^2). \end{aligned}$$

Resolve into factors:

21. $c(a+b) + d(a+b).$

22. $p(p-q) + q(p-q).$

23. $7m(x-1) + 2(x-1).$

24. $2a(x^2+x+1) + 3b(x^2+x+1).$

25. $3p(ax+b) + q(ax+b) + 2r(ax+b).$

26. $m^2(pq+r) + mn(pq+r) + n^2(pq+r).$

27. $a(x-y) + b(y-x) + (c+d)(x-y).$

28. $(x^2+7x)^2 - 25(x^2+7x).$

29. $15(p^2-qr)^3 - 10(p^2-qr)^2$

30. $(2p+3q)(x-y) + (p-2q)(x-y) + (3p+q)(x-y).$

31. $(x+y)^3 - 3xy(x+y).$

2. Type II. Factors of expressions of the form

$$ac + bc + ad + bd.$$

We notice that no factor is common to all the terms, but c is common to the first two terms and d is common to the last two terms. Enclosing the first two terms in one pair of brackets and the last two in another pair, we have

$$\begin{aligned} ac + bc + ad + bd \\ = (ac + bc) + (ad + bd) \\ = c(a+b) + d(a+b) \\ = (a+b)(c+d). \end{aligned}$$

Example 1. Resolve into factors $3ab + 6bc + 4ad + 8cd.$

$$\begin{aligned} 3ab + 6bc + 4ad + 8cd \\ = (3ab + 6bc) + (4ad + 8cd) \\ = 3b(a+2c) + 4d(a+2c) \\ = (a+2c)(3b+4d). \end{aligned}$$

Example 2. Factorise $3x^2 + 4yz - 4xy - 3xz$.

Re-arranging the terms, the expression

$$= 3x^2 - 3xz - 4xy + 4yz$$

$$= 3x(x - z) - 4y(x - z)$$

$$= (x - z)(3x - 4y).$$

EXERCISE 33.

Factorise :

- | | |
|--------------------------------|-----------------------------------|
| 1. $5x + 5y + ax + ay.$ | 2. $ax - ay - bx + by.$ |
| 3. $mx - 3y - my + 3x.$ | 4. $xy + 3x + 2y + 6.$ |
| 5. $x^2 - 3ax + bx - 3ab.$ | 6. $a^3 + 3a^2 + 9a + 27.$ |
| 7. $3p^3 - 5p^2 + 6p - 10.$ | 8. $6p^2 - 9pq + 4pr - 6qr.$ |
| 9. $pq + p + q + 1.$ | 10. $a^3 + a^2 - a - 1.$ |
| 11. $2a^3 - 5a^2 + 2a - 5.$ | 12. $5a^3 - 2a^2 + 5a - 2.$ |
| 13. $x^2 - xy - 3x + 3y.$ | 14. $6a^2 + 3ab - 2ac - bc.$ |
| 15. $2a^4 - a^3 + 4a - 2.$ | 16. $y^3 - y^2 - ay + y + a - 1.$ |
| 17. $11x^3 + 7x + 35 + 55x^2.$ | 18. $ax^3 + b + bx^2 + ax.$ |
| 19. $6xy + 6 - 9y - 4x.$ | 20. $x^2 - 2ax + 2ab - bx.$ |

Sometimes we have to remove the brackets and re-arrange the terms, as in the next example.

Example 3. $a^3x - a^2b(x - y) - ab^2(y - z) - b^3z.$

$$\begin{aligned} \text{The expression} &= a^3x - a^2bx + a^2by - ab^2y + ab^2z - b^3z \\ &= (a^3x - a^2bx) + (a^2by - ab^2y) + (ab^2z - b^3z) \\ &= a^2x(a - b) + aby(a - b) + b^2z(a - b) \\ &= (a - b)(a^2x + aby + b^2z). \end{aligned}$$

Resolve into factors :

21. $ax^2 + (a - 1)x - 1.$
22. $abx^2 + (ay - b)x - y.$
23. $a^3x + a^2(x - y) - a(y + z) - z.$
24. $a(x^2 - 1) + x^2(bx - c) + x(cx^3 - b).$
25. $x^3(kx + l) - x(kx + l) + m(x^2 - 1).$
26. $(a - 2b)^2 + 3a - 6b.$
27. $ab(c^2 + 1) + c(a^2 + b^2).$
28. $ab(x^2 + y^2) - xy(a^2 + b^2).$

3. Type III. Factors of expressions of the form

$$a^2 + 2ab + b^2 \text{ or } a^2 - 2ab + b^2.$$

Example 1. Factorise $x^2 + 14x + 49$.

$$\begin{aligned} x^2 + 14x + 49 &= x^2 + 2 \cdot x \cdot 7 + 7^2 \\ &= (x + 7)^2. \end{aligned}$$

Example 2. Factorise $36a^2 - 12a + 1$.

$$\begin{aligned} 36a^2 - 12a + 1 &= (6a)^2 - 2 \cdot (6a) \cdot 1 + 1^2 \\ &= (6a - 1)^2. \end{aligned}$$

Example 3. Factorise $9 + 42xy + 49x^2y^2$.

$$\begin{aligned} 9 + 42xy + 49x^2y^2 &= 3^2 + 2 \cdot 3 \cdot (7xy) + (7xy)^2 \\ &= (3 + 7xy)^2. \end{aligned}$$

EXERCISE 34. [Mainly oral]

Resolve into factors :

- | | |
|---|---|
| 1. $4a^2 + 28a + 49$. | 2. $25p^2 - 30p + 9$. |
| 3. $25a^2 + 40ab + 16b^2$. | 4. $49m^2 - 56mn + 16n^2$. |
| 5. $36a^2 + 60ab + 25b^2$. | 6. $36a^2 - 84ab + 49b^2$. |
| 7. $49a^2 + 42ab + 9b^2$. | 8. $25x^2 - 60xy + 36y^2$. |
| 9. $16x^2 + 56xy + 49y^2$. | 10. $81a^2 - 126ab + 49b^2$. |
| 11. $81x^2 + 90xy + 25y^2$. | 12. $9m^2 - 66mn + 121n^2$. |
| 13. $64x^2 + 16xy + y^2$. | 14. $16p^2 - 48pq + 36q^2$. |
| 15. $9x^2 + 54xy + 81y^2$. | 16. $25m^2 - 70mn + 49n^2$. |
| 17. $x^2 + 2 + \frac{1}{x^2}$. | 18. $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2}$. |
| 19. $a^2x^2 + 6abxy + 9b^2y^2$. | 20. $16a^2b^2 - 8abc^2 + c^4$. |
| 21. $(2a + 3b)^2 + 2(2a + 3b)(c + d) + (c + d)^2$. | |
| 22. $(3p - q)^2 - 2(3p - q)(r + s) + (r + s)^2$. | |

4. Type IV. Factors of expressions of the form $a^2 - b^2$.

Example 1. Resolve into factors $16a^2 - 25b^2$.

$$\begin{aligned} 16a^2 - 25b^2 &= (4a)^2 - (5b)^2 \\ &= (4a + 5b)(4a - 5b). \end{aligned}$$

NOTE. The first step, *i.e.*, the reduction of the given expression to the standard form, is essential.

Example 2. Resolve into factors $1 - 49x^2$.

$$\begin{aligned} 1 - 49x^2 &= (1)^2 - (7x)^2 \\ &= (1 + 7x)(1 - 7x). \end{aligned}$$

EXERCISE 35.

Resolve into factors (*mentally*):

- | | | |
|--------------------|----------------------|---------------------|
| 1. $16x^2 - 9$. | 2. $81a^2 - 25b^2$. | 3. $1 - 36x^2$. |
| 4. $64a^2 - 1$. | 5. $9x^2 - 100$. | 6. $49 - 25a^2$. |
| 7. $121 - c^2$. | 8. $1 - 49a^2c^2$. | 9. $a^4 - 25$. |
| 10. $16x^6 - 49$. | 11. $1 - 81x^4y^2$. | 12. $x^6 - 36y^2$. |

Example 3. Resolve into factors $16x^5 - x$.

$$\begin{aligned} 16x^5 - x &= x(16x^4 - 1) \\ &= x \{ (4x^2)^2 - (1)^2 \} \\ &= x(4x^2 + 1)(4x^2 - 1) \\ &= x(4x^2 + 1) \{ (2x)^2 - (1)^2 \} \\ &= x(4x^2 + 1)(2x + 1)(2x - 1). \end{aligned}$$

Resolve into factors :

- | | |
|----------------------------------|--------------------------------|
| 13. $81x^4 - 1$. | 14. $16x^5 - 81x$. |
| 15. $1 - 16x^4$. | 16. $a^2 - 81a^6$. |
| 17. $36x^7 - 25x^3a^4$. | 18. $192x^9 - 243x^5y^4$. |
| 19. $484a^{17}b^9 - 324a^5b^3$. | 20. $(2x - 3y)^3 - 8x + 12y$. |

Example 4. Resolve into factors $4(a - b)^2 - 9(c - d)^2$

$$\begin{aligned} 4(a - b)^2 - 9(c - d)^2 &= 2^2(a - b)^2 - 3^2(c - d)^2 \\ &= (2a - 2b)^2 - (3c - 3d)^2 \\ &= (2a - 2b + 3c - 3d)(2a - 2b - 3c + 3d). \end{aligned}$$

Resolve into factors :

- | | |
|---|-----------------------------------|
| 21. $(a + 5b)^2 - 49c^2$. | 22. $x^2 - (2y - 3z)^2$. |
| 23. $(a + b)^2 - (a - b)^2$. | 24. $(3x - 2y)^2 - (2x + y)^2$. |
| 25. $16(x - y)^2 - 9(x + y)^2$. | 26. $49(a + b)^2 - 36(a - b)^2$. |
| 27. $81(m + 2n)^2 - 25(2m - n)^2$. | |
| 28. $4(3m - 2n)^2 - 9(2m - 3n)^2$. | 29. $(8a + 5)^2 - (2a - 7)^2$. |
| 30. $(a + b - c)^2 - (a - b + c)^2$. | |
| 31. $16(a + 3b - 4c)^2 - 9(2a - b + 3c)^2$. | |
| 32. $(3a^2 - 5a + 7)^2 - (3a^2 - 5a - 7)^2$. | |
| 33. $(x + a)^4 - (x - a)^4$. | |

Simplify :

34. $2396^2 - 2391^2$. 35. $857 \times 857 - 143 \times 143$.

36. $\cdot 738 \times \cdot 738 - \cdot 262 \times \cdot 262$.

5. **Type V.** Factors of expressions of the form

$$a^3 + b^3 \text{ or } a^3 - b^3.$$

Example 1. Resolve into factors $8a^3 + 27b^3$.

$$\begin{aligned} 8a^3 + 27b^3 &= (2a)^3 + (3b)^3 \\ &= (2a + 3b) \{ (2a)^2 - (2a)(3b) + (3b)^2 \} \\ &= (2a + 3b)(4a^2 - 6ab + 9b^2). \end{aligned}$$

NOTE. The first step, *i.e.* the reduction of the given expression to the standard form, is essential.

Example 2. Resolve into factors $27x^3 - 1$.

$$\begin{aligned} 27x^3 - 1 &= (3x)^3 - (1)^3 \\ &= (3x - 1) \{ (3x)^2 + (3x)(1) + (1)^2 \} \\ &= (3x - 1)(9x^2 + 3x + 1). \end{aligned}$$

EXERCISE 36.

Resolve into factors (*mentally*) :

- | | | |
|------------------------|----------------------|----------------------------|
| 1. $m^3 + 1$. | 2. $1 - a^3$. | 3. $8 - x^3$. |
| 4. $p^3 + 64$. | 5. $125a^3 - 8$. | 6. $p^3q^3 + 216$. |
| 7. $64a^3 + 1000$. | 8. $729x^3 - 216$. | 9. $125x^6 + 64y^3$. |
| 10. $8x^3 - 1000b^3$. | 11. $64m^3n^3 + 1$. | 12. $729x^3y^3 - 512z^3$. |

Example 3. Resolve into factors $27a^7 - ab^6$

$$\begin{aligned} 27a^7 - ab^6 &= a(27a^6 - b^6) \\ &= a \{ (3a^2)^3 - (b^2)^3 \} \\ &= a(3a^2 - b^2) \{ (3a^2)^2 + (3a^2)(b^2) + (b^2)^2 \} \\ &= a(3a^2 - b^2)(9a^4 + 3a^2b^2 + b^4). \end{aligned}$$

Resolve into factors :

- | | |
|---------------------------------------|---------------------------------------|
| 13. $a^6 - 8b^6$. | 14. $27m^6 + 125n^6$. |
| 15. $a^9 - b^9$. | 16. $a^{12} - 8b^6c^6$. |
| 17. $343a^{13}b + 64ab^{13}$. | 18. $(3x + 2y)^3 - z^3$. |
| 19. $(2a - 3b)^3 + 27c^3$. | 20. $(2a + b)^3 - (a + 2b)^3$. |
| 21. $(a - 2b)^3 + 1$. | 22. $x^3 - 3x^2y + 3xy^2 - y^3 - 1$. |
| 23. $125(a - 2b)^3 - 27(3a - 4b)^3$. | 24. $64(x - 3y)^3 + 343(2x + y)^3$. |

Simplify :

$$25. \quad 12^3 + 8^3. \quad 26. \quad 25^3 - 5^3. \quad 27. \quad 32^3 - 12^3.$$

Example 4. Resolve into factors $x^6 - 64$.

$$\begin{aligned} x^6 - 64 &= (x^3)^2 - (8)^2 \\ &= (x^3 + 8)(x^3 - 8) \\ &= (x + 2)(x^2 - 2x + 4)(x - 2)(x^2 + 2x + 4). \end{aligned}$$

NOTE. If you do this example by reducing the expression to the form $(x^2)^3 - (4)^3$ and then factorise it, it will be found that the method given above is more convenient than this one.

Resolve into factors :

$$\begin{aligned} 28. \quad a^6 - b^6. \quad 29. \quad a^{12} - b^{12}. \quad 30. \quad 64a^{12} - 1. \\ 31. \quad 729a^6 - b^6. \quad 32. \quad a^7 - ab^6. \quad 33. \quad 3x^{13} - 192x^7y^6. \end{aligned}$$

6. Type VI. Factors of quadratic expressions of the form $x^2 + px + q$.

Analysis. Since the first term of $x^2 + px + q$ is x^2 , therefore the first term of each factor will be x .

Let $(x + m)(x + n)$ be the factors of $x^2 + px + q$.

Now, $(x + m)(x + n)$ will be the factors of $x^2 + px + q$ if $x^2 + (m + n)x + mn \equiv x^2 + px + q$.

or, if $m + n = +p$ and $mn = +q$.

Thus, in order to find out the 2nd term in each of the above factors, we have to think out two numbers

- (i) whose product may be equal to $(+q)$ and
- (ii) whose algebraic sum may be equal to $(+p)$.

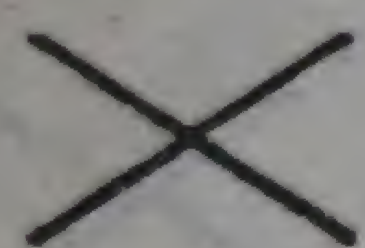
The application of these principles is illustrated in the following examples :

Example 1. Resolve into factors $x^2 + 11x + 24$.

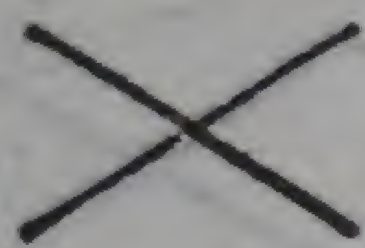
Obviously the first term of each factor will be x and the 2nd terms will be such that they are factors of $+24$ and their sum $= +11$.

The pairs of possible factors in which the product of the second terms = +24 are :

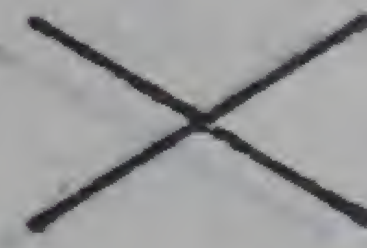
$$(i) x + 1 \quad (ii) x - 1 \quad (iii) x + 2 \quad (iv) x - 2$$



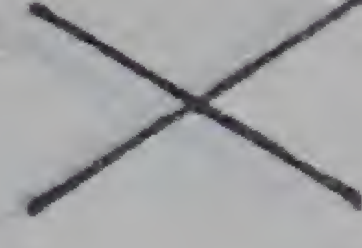
$$x + 24$$



$$x - 24$$

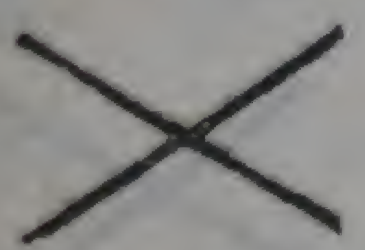


$$x + 12$$



$$x - 12$$

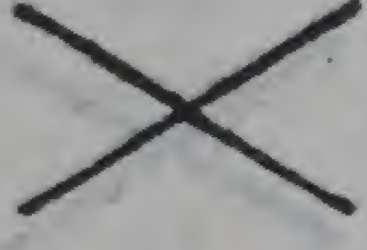
$$(v) x + 3 \quad (vi) x - 3 \quad (vii) x + 4 \quad (viii) x - 4$$



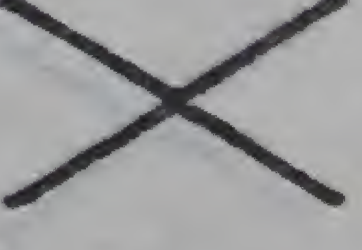
$$x + 8$$



$$x - 8$$



$$x + 6$$



$$x - 6$$

We notice that it is only in (v) that the sum of such factors = +11,

$\therefore x + 3$ and $x + 8$ are the required factors.

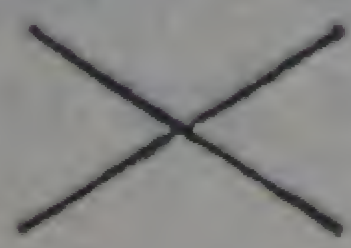
Example 2. Resolve into factors $x^2 - 11x + 24$.

Here we have to find two numbers whose product = +24 and whose sum = -11.

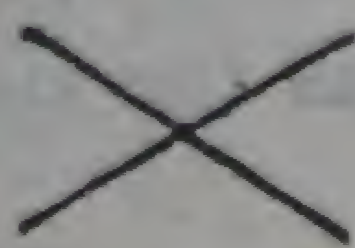
Since the product is positive, the required numbers will be either both positive or both negative, and as their sum is negative, therefore both the required numbers will be negative.

Hence the pairs of possible factors are :

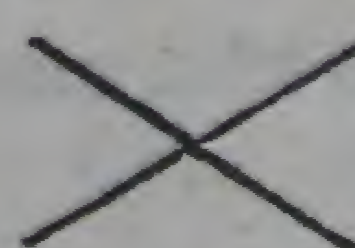
$$(i) x - 1 \quad (ii) x - 2 \quad (iii) x - 3 \quad (iv) x - 4$$



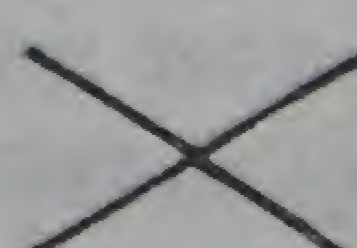
$$x - 24$$



$$x - 12$$



$$x - 8$$



$$x - 6.$$

We notice that it is only in (iii) that the sum of 2nd terms is -11,

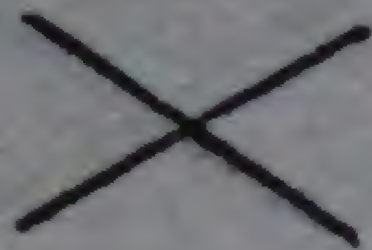
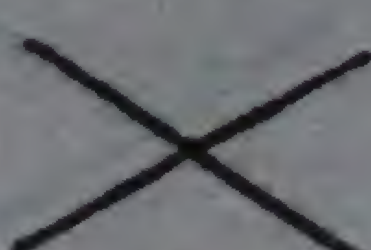
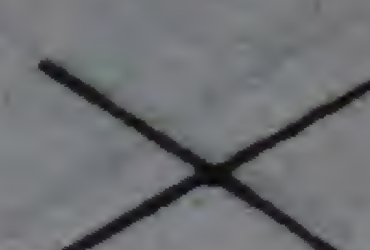
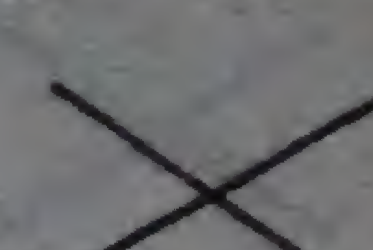
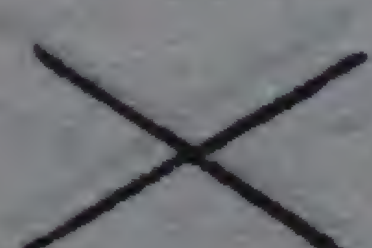
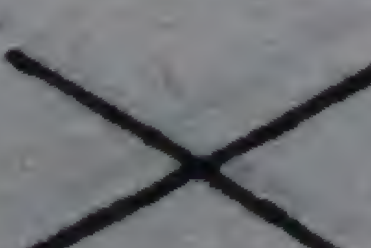
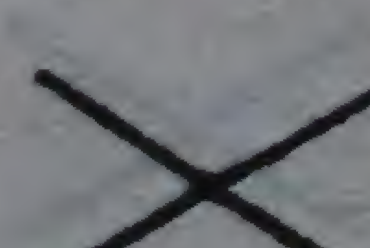
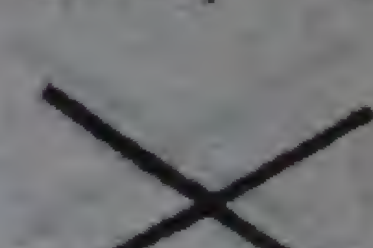
$\therefore x - 3$ and $x - 8$ are the required factors.

Example 3. Resolve into factors $x^2 + 5x - 24$.

Here we have to find two numbers whose product = -24 and whose sum = +5.

Since the product = -24, one of the numbers must be negative and the other positive.

The pairs of the possible factors in which the product of the 2nd terms = -24 are ;

(i) $x - 1$  $x + 24$	(ii) $x + 1$  $x - 24$	(iii) $x - 2$  $x + 12$	(iv) $x + 2$  $x - 12$
(v) $x - 3$  $x + 8$	(vi) $x + 3$  $x - 8$	(vii) $x - 4$  $x + 6$	(viii) $x + 4$  $x - 6.$

We notice that it is only in (v) that the sum of the 2nd terms = $+5$,

$\therefore x - 3$ and $x + 8$ are the required factors.

Example 4. Resolve into factors $x^2 - 5x - 24$.

Here we have to find two numbers whose product = -24 and whose sum = -5 .

Since the product = -24 , one of the numbers must be negative and the other positive.

Proceeding as in example 3, we find that it is only in (vi) that the sum of the 2nd terms = -5 .

$\therefore x + 3$ and $x - 8$ are the required factors.

NOTE.—(i) In types of examples 3 and 4, it is worth noting that the greater number has the same sign as the sign of the 2nd term in the original expression.

(ii) After some practice, it is not difficult to attempt such exercises *mentally*.

EXERCISE 37.

Resolve into factors :

1. $x^2 + 13x + 42.$

2. $a^2 - 18a + 65.$

3. $x^2 + 18x + 65.$

4. $p^2 - 15p + 54.$

5. $x^2 + 18x + 72.$

6. $x^2 + 20x + 91.$

7. $x^2 - 22x + 117.$

8. $x^2 + 21x + 104.$

9. $a^2 + 5a - 104.$

10. $x^2 + 21x + 108.$

- | | |
|------------------------|-------------------------|
| 11. $a^2 + 3a - 108.$ | 12. $x^2 + 27x + 180.$ |
| 13. $p^2 + p - 156.$ | 14. $a^2 - 3a - 180.$ |
| 15. $a^2 - 2a - 143.$ | 16. $a^2 - a - 240.$ |
| 17. $x^2 + 8x - 105.$ | 18. $a^2 - 26a - 120.$ |
| 19. $1 - 19x + 60x^2.$ | 20. $1 - 49a - 102a^2.$ |

Resolve into factors (*mentally*):

- | | |
|-------------------------|------------------------|
| 21. $x^2 + 21x + 98.$ | 22. $a^2 - 17a + 72.$ |
| 23. $p^2 - 5p - 176.$ | 24. $m^2 + 6m - 91.$ |
| 25. $p^2 - 10p + 24.$ | 26. $x^2 + 7x - 30.$ |
| 27. $m^2 + 9m - 36.$ | 28. $x^2 - 11x - 80.$ |
| 29. $a^2 - 10a - 56.$ | 30. $m^2 + 5m - 84.$ |
| 31. $x^2 - 20x + 96.$ | 32. $x^2 - 6x - 72.$ |
| 33. $x^2 - 25x + 84.$ | 34. $x^2 + 17x - 84.$ |
| 35. $a^2 - 9a - 90.$ | 36. $x^2 - 22x - 48.$ |
| 37. $x^2 - x - 42.$ | 38. $a^2 + a - 72.$ |
| 39. $1 - 19a - 120a^2.$ | 40. $1 + 23x - 78x^2.$ |

Example 5. Resolve into factors $x^2 - 2xy - 15y^2$.

The factors will evidently be $(x + py)$, and $(x + qy)$, where p and q are such that their sum $= -2$ and their product $= -15$.

Proceeding as before, it is easy to see that -5 and $+3$ are the numbers whose sum $= -2$ and whose product $= -15$.

$$\therefore x^2 - 2xy - 15y^2 = (x - 5y)(x + 3y)$$

Resolve into factors :

- | | |
|---------------------------|---------------------------|
| 41. $p^2 - 12pq + 20q^2.$ | 42. $m^2 - 12mn + 32n^2.$ |
| 43. $a^2 + ab - 30b^2.$ | 44. $a^2 - 4ab - 45b^2.$ |
| 45. $p^2 - 13pq - 48q^2.$ | 46. $a^2 + 20ab + 96b^2.$ |
| 47. $m^2 + 3mn - 28n^2.$ | 48. $x^2 - 11xy - 80y^2.$ |
| 49. $a^2 - 6ab - 91b^2.$ | 50. $x^2 - 6xy - 135y^2.$ |

Example 6. Resolve into factors $x^4 + x^2 - 20$.

Putting a for x^2 , the given expression becomes

$$a^2 + a - 20, \text{ and when factorised } = (a + 5)(a - 4).$$

$$\text{Hence } x^4 + x^2 - 20 = (x^2 + 5)(x^2 - 4)$$

$$= (x^2 + 5)(x + 2)(x - 2).$$

Example 7. Resolve into factors $(a^2 + 2a)^2 - 4(a^2 + 2a) - 21$.

Putting x for $a^2 + 2a$, the given expression becomes $x^2 - 4x - 21$, and when factorised $= (x - 7)(x + 3)$.

Hence the given expression $= (a^2 + 2a - 7)(a^2 + 2a + 3)$.

Example 8. Resolve into factors

$$(2a + b)^2 + (2a + b)(a - b) - 12(a - b)^2.$$

Putting x for $2a + b$ and y for $a - b$, the given expression becomes $x^2 + xy - 12y^2$, and when factorised

$$= (x + 4y)(x - 3y).$$

Hence the given expression

$$\begin{aligned} &= \{ (2a + b) + 4(a - b) \} \{ 2a + b - 3(a - b) \} \\ &= (2a + b + 4a - 4b)(2a + b - 3a + 3b) \\ &= (6a - 3b)(-a + 4b) \\ &= -3(2a - b)(a - 4b). \end{aligned}$$

Resolve into factors :

51. $a^4 - a^2 - 12.$

52. $x^4 + 3x^2 - 28.$

53. $x^4 - 18x^2 - 175.$

54. $a^4 - 25a^2 + 136.$

55. $x^4 - 2x^2y^2 - 63y^4.$

56. $a^6 + 2a^3 - 3.$

57. $p^6 + 26p^3 - 27.$

58. $x^6 - 27x^3 + 180.$

59. $m^8 - 11m^4n^4 - 80n^8.$

60. $p^{12} - 7p^6q^6 - 8q^{12}.$

61. $(a^2 - 2a)^2 - (a^2 - 2a) - 30.$

62. $(x^2 + x)^2 - 2(x^2 + x) - 35.$

63. $(m^2 - 3m)^2 - 8(m^2 - 3m) + 12.$

64. $(p^2 - 5p)^2 + 5(p^2 - 5p) - 36.$

65. $(a^2 - 4a)^2 + 16(a^2 - 4a) + 48.$

66. $(x^2 - 6x)^2 - 3(x^2 - 6x) - 180.$

67. $(m^2 + 7m)^2 - (m^2 + 7m) - 156.$

68. $(p^2 + 4p)^2 + 21(p^2 + 4p) + 98.$

69. $(x^2 - 8x)^2 - (x^2 - 8x) - 240.$

70. $(m^2 - 7m)^2 - 2(m^2 - 7m) - 195.$

71. $(x + y)^2 + 7(x + y)(x - y) + 12(x - y)^2.$

72. $(x + 2y)^2 - 2(x + 2y)(x - y) - 15(x - y)^2.$

73. $(2a-3b)^2 - 11(2a-3b)(a+2b) + 24(a+2b)^2$.
 74. $(3m-n)^2 + 3(3m-n)(m+n) - 28(m+n)^2$.
 75. $(3a+4b)^2 - 3(3a+4b)(a-2b) - 10(a-2b)^2$.
 76. $(4m-3n)^2 - 2(4m-3n)(m+2n) - 35(m+2n)^2$.
 77. $(2a-5b)^2 - 5(2a-5b)(a-3b) - 6(a-3b)^2$.
 78. $(a^2-ab)^2 - 2(a^2-ab)(ab-b^2) - 8(ab-b^2)^2$.
 79. $(2a^2+3b^2)^2 + 5(2a^2+3b^2)(a^2-2b^2) - 14(a^2-2b^2)^2$.
 80. $(a^2-4ab)^2 - 9(a^2-4ab)(ab-4b^2) + 18(ab-4b^2)^2$.

Simplify by factors :

81. $\frac{x^2+7x+10}{x^2+5x}$.

82. $\frac{x^2+2x+1}{x^2+5x+4}$.

83. $\frac{x^2-4x-5}{x^2-2x-15} \div \frac{x^2-x-2}{x^2-x-12}$.

84. What is the quotient if $x^2+7x+12$ is divided by $x+4$?

85. If $x+2$ is a factor of $x^2+mx+14$, what is the value of m ?

86. If $x-3$ is a factor of $x^2-px-15$, what is the value of p ?

87. If $x+4$ is a factor of x^2+9x+c , what is the value of c ?

7. Type VII. Factors of expressions of the form ax^2+bx+c .

Analysis. $ax^2+bx+c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$.

Let $(x+p)(x+q)$ be the factors of $x^2 + \frac{b}{a}x + \frac{c}{a}$.

Here p, q must be equal to $\frac{c}{a}$

and $p+q$ „ „ „ „ $\frac{b}{a}$

or $a(p \cdot q)$ „ „ „ „ c

i.e., $(ap) \cdot (aq)$ „ „ „ „ ac

and $ap+aq$ „ „ „ „ b .

In other words, we have to find out (ap) and (aq) such two numbers whose product $= ac$ and whose algebraic sum $= b$.

The application of these principles is illustrated in the following examples :

Example 1. Resolve into factors $5x^2 + 33x - 14$.

The *first* terms of the factors must be $5x$ and x .

The *second* „ „ „ have different signs and are the factors of -14 .

The possible pairs of factors are

- | | | |
|-------|---|---|
| (i) | $\left. \begin{array}{l} 5x + 1 \\ \times \\ x - 14 \end{array} \right\}$ | co-eff. of x would be $-70 + 1 = -69$. |
| (ii) | $\left. \begin{array}{l} 5x - 1 \\ \times \\ x + 14 \end{array} \right\}$ | „ „ „ $+70 - 1 = +69$. |
| (iii) | $\left. \begin{array}{l} 5x + 14 \\ \times \\ x - 1 \end{array} \right\}$ | „ „ „ $-5 + 14 = +9$. |
| (iv) | $\left. \begin{array}{l} 5x - 14 \\ \times \\ x + 1 \end{array} \right\}$ | „ „ „ $+5 - 14 = -9$. |
| (v) | $\left. \begin{array}{l} 5x + 2 \\ \times \\ x - 7 \end{array} \right\}$ | „ „ „ $-35 + 2 = -33$. |
| (vi) | $\left. \begin{array}{l} 5x - 2 \\ \times \\ x + 7 \end{array} \right\}$ | „ „ „ $+35 - 2 = +33$. |

$$(vii) \left. \begin{array}{r} 5x + 7 \\ \times \\ x - 2 \end{array} \right\} \text{co-eff. of } x \text{ would be } -10 + 7 = -3.$$

$$(viii) \left. \begin{array}{r} 5x - 7 \\ \times \\ x + 2 \end{array} \right\} \text{co-eff. of } x \text{ would be } +10 - 7 = +3.$$

As in case (vi) alone the co-eff. of $x = +33$, therefore the required factors are $5x - 2$ and $x + 7$.

Example 2. Resolve into factors $8x^2 - 6xy - 9y^2$.

Here we have to find two numbers whose product $= -72$ and whose algebraic sum $= -6$.

On trial, we find that -12 and $+6$ are the numbers which satisfy the above conditions.

$$\begin{aligned} \therefore 8x^2 - 6xy - 9y^2 &= 8x^2 - 12xy + 6xy - 9y^2 \\ &= (8x^2 - 12xy) + (6xy - 9y^2) \\ &= 4x(2x - 3y) + 3y(2x - 3y) \\ &= (2x - 3y)(4x + 3y). \end{aligned}$$

Another Method.

When the factors cannot be found so easily, the following method may be employed:

Example 3. Resolve into factors $18x^2 - 9x - 2$.

$$18x^2 - 9x - 2 = \frac{1}{18} \left\{ (18x)^2 - 9(18x) - 36 \right\}$$

(Writing y for $18x$)

$$= \frac{1}{18} \left\{ y^2 - 9y - 36 \right\} = \frac{1}{18} (y - 12)(y + 3)$$

$$= \frac{1}{18} (18x - 12)(18x + 3)$$

$$= \left(\frac{18x - 12}{6} \right) \left(\frac{18x + 3}{3} \right)$$

$$= (3x - 2)(6x + 1).$$

EXERCISE 38.

Resolve into factors :

- | | |
|----------------------------|-----------------------------|
| 1. $2x^2 + 11x + 14.$ | 2. $3x^2 + 14x - 5.$ |
| 3. $5x^2 + 8x + 3.$ | 4. $6x^2 + 5x - 6.$ |
| 5. $6x^2 - 4x - 2.$ | 6. $8x^2 - 14x + 3.$ |
| 7. $8x^2 + 2x - 3.$ | 8. $12m^2 + 7m - 10.$ |
| 9. $14a^2 + 29a + 15.$ | 10. $20p^2 + 44p - 15.$ |
| 11. $3 - 5p + 2p^2.$ | 12. $2 + 7m - 15m^2.$ |
| 13. $9a^2 - 15ab + 4b^2.$ | 14. $2m^2 - mn - 21n^2.$ |
| 15. $4p^2 + 5pq - 9q^2.$ | 16. $15x^2 - 77xy + 10y^2.$ |
| 17. $12a^2 + 28ab - 5b^2.$ | 18. $10x^2 - 41xy + 21y^2.$ |

Simplify by factors :

19. $\frac{6x^2 + x - 35}{12x^2 - 19x - 21} \times \frac{4x^2 - 5x - 6}{2x^2 - x - 15}.$
20. $\frac{6x^2 - 7x - 20}{3x^2 - 14x - 24} \div \frac{21x^2 + 41x + 10}{7x^2 - 40x - 12}.$

Resolve into factors :

21. $2(x+y)^2 - 3z(x+y) - 2z^2.$ [Hint. Put a for $(x+y).$]

22. $2(a^2 + b^2)^2 + 5ab(a^2 + b^2) + 2a^2b^2.$

23. $x^2(x+1)^2 - 14(x+1)x + 24.$

24. $8(a+1)^2 + 2(a+1)(b+2) - 15(b+2)^2.$

[Hint. Put x for $(a+1)$ and y for $(b+2).$]

25. $8(3m - 4n)^2 - 6(3m - 4n)(4m - 3n) - 35(4m - 3n)^2.$

26. $6x^4 - 7x^2 - 20.$

27. $6x^4 - 7x^2 - 3.$

28. $5x^6 - 7x^3 - 6.$

29. $18x^4 + 25x^2y^2 - 3y^4.$

30. $8x^6 - 65x^3 + 8.$

31. $4x^8 - 17x^4y^4 + 4y^8.$

32. $5x^2 - 13xy + 6y^2.$

33. $5x^2 - 13xy - 6y^2.$

34. $7x^2 - 25x + 12.$

35. $7x^2 - 25x - 12.$

36. $x^3y^3 - 9x^2y^2 + 20xy.$

37. $8a^3 - 2a^2b - 15ab^2.$

38. $(a+7)(a-10) + 16.$

39. $(x^2 - 4x)(x^2 - 4x - 1) - 20.$ [Hint. Put a for $(x^2 - 4x).$]

40. $(x^2 + x - 6)(x^2 + x - 20) - 15.$

41. $(x^2 + 8x + 7)(x^2 + 8x + 15) - 9.$

42. $\frac{x^2}{4} + 2x - 5.$ [Hint. Put a for $\frac{x}{2}$.]

8. **Type VIII.** Factors of expressions of the form

$$a^3 + b^3 + c^3 - 3abc.$$

By a well-known formula, $a^3 + b^3 + c^3 - 3abc$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc). \quad (i)$$

As $a^2 + b^2 + c^2 - ab - ac - bc$

$$\begin{aligned} &= \frac{1}{2} \{ 2a^2 + 2b^2 + 2c^2 - 2ab - 2ac - 2bc \} \\ &= \frac{1}{2} \{ (a^2 - 2ab + b^2) + (b^2 - 2bc + c^2) + (c^2 - 2ac + a^2) \} \\ &= \frac{1}{2} \{ (a - b)^2 + (b - c)^2 + (c - a)^2 \} \end{aligned}$$

\therefore the formula can be put in the following form :

$$a^3 + b^3 + c^3 - 3abc = \frac{1}{2}(a + b + c) \{ (a - b)^2 + (b - c)^2 + (c - a)^2 \}. \quad (ii)$$

Example 1. Resolve into factors $8a^3 - 27b^3 + 1 + 18ab$.

Reducing the given expression to the standard form, we have

$$\begin{aligned} &(2a)^3 + (-3b)^3 + (1)^3 - 3(2a)(-3b)(1) \\ &= (2a - 3b + 1) \{ (2a)^2 + (-3b)^2 + (1)^2 - (2a)(-3b) \\ &\quad - (2a)(1) - (-3b)(1) \} \\ &= (2a - 3b + 1)(4a^2 + 9b^2 + 1 + 6ab - 2a + 3b). \end{aligned}$$

Example 2. Factorise $(x + y)^3 + (y + z)^3 + (z + x)^3 - 3(x + y)(y + z)(z + x).$

Putting a for $(x + y)$, b for $(y + z)$ and c for $(z + x)$, we get

$$\begin{aligned} &a^3 + b^3 + c^3 - 3abc \\ &= \frac{1}{2}(a + b + c) \{ (a - b)^2 + (b - c)^2 + (c - a)^2 \}. \end{aligned}$$

Substituting the values of a , b and c , we get the expression

$$\begin{aligned} &= \frac{1}{2} \{ (x + y) + (y + z) + (z + x) \} \{ (x - z)^2 + (y - x)^2 \\ &\quad + (z - y)^2 \} \\ &= (x + y + z)(2x^2 + 2y^2 + 2z^2 - 2xy - 2xz - 2yz) \\ &= 2(x + y + z)(x^2 + y^2 + z^2 - xy - xz - yz). \end{aligned}$$

EXERCISE 39.

Factorise :

1. $a^3 - 8b^3 + 27c^3 + 18abc$. 2. $x^3 - y^3 - z^3 - 3xyz$.
3. $a^3 - b^3 - 1 - 3ab$. 4. $x^3 - 3xy + y^3 + 1$.
5. $a^3 + b^3 + 18ab - 216$. 6. $x^3 - 8y^3 - 24xy - 64$.
7. $64a^3 - 27b^3 + 1 + 36ab$. 8. $27x^3 - 125y^3 - 180xy - 64$.
9. $8a^3 - 27b^3 - c^3 - 18abc$.
10. $x^3 - \frac{1}{x^3} - 36$. [Hint. $-36 = -27 - 9$.]
11. $a^6 + 32a^3 - 64$. [Hint. $32a^3 = 8a^3 + 24a^3$.]
12. $(x+y)^3 + (y+z)^3 - (z+x)^3 + 3(x+y)(y+z)(z+x)$.
13. $(a-b)^3 - (b-c)^3 + (c-a)^3 + 3(a-b)(b-c)(c-a)$.
14. $(x+1)^3 + (x+2)^3 + (x+3)^3 - 3(x+1)(x+2)(x+3)$.
15. $(x+y+2z)^3 + (y+z+2x)^3 + (z+x+2y)^3$
 $- 3(x+y+2z)(y+z+2x)(z+x+2y)$.

9. **Type IX.** Factors of expressions of the form
 $(x+a)(x+b)(x+c)(x+d) + k$.

Example 1. Factorise $(x+1)(x+3)(x-4)(x-6) + 13$.

$$\begin{aligned} \text{The expression} &= \{ (x+1)(x-4) \} \{ (x+3)(x-6) \} + 13 \\ &= (x^2 - 3x - 4)(x^2 - 3x - 18) + 13. \end{aligned}$$

Putting a for $x^2 - 3x$, we get the expression

$$\begin{aligned} &= (a-4)(a-18) + 13 \\ &= a^2 - 22a + 72 + 13 \\ &= a^2 - 22a + 85 \\ &= (a-5)(a-17). \end{aligned}$$

Substituting $x^2 - 3x$ for a , the expression

$$= (x^2 - 3x - 5)(x^2 - 3x - 17).$$

Example 2. Factorise $(a+4)(a-6)(a+2)(a-3) - 6a^2$.

$$\begin{aligned} \text{The expression} &= \{ (a+4)(a-3) \} \{ (a-6)(a+2) \} - 6a^2 \\ &= (a^2 + a - 12)(a^2 - 4a - 12) - 6a^2 \\ &= (a^2 - 12 + a)(a^2 - 12 - 4a) - 6a^2. \end{aligned}$$

Putting x for $a^2 - 12$, the expression

$$= (x + a)(x - 4a) - 6a^2$$

$$= x^2 - 3ax - 4a^2 - 6a^2$$

$$= x^2 - 3ax - 10a^2$$

$$= (x - 5a)(x + 2a).$$

Substituting $a^2 - 12$ for x , the expression

$$= (a^2 - 5a - 12)(a^2 + 2a - 12).$$

NOTE. In factorising expressions of this type, we take the four factors in the expression in two pairs so that their products may contain the same *first two terms*, as illustrated in the above two examples.

EXERCISE 40.

Factorise :

1. $(x + 1)(x + 2)(x + 3)(x + 4) - 3.$
2. $(x - 1)(x - 2)(x + 3)(x + 4) + 4.$
3. $(x + 2)(x + 3)(x - 5)(x - 6) + 12.$
4. $(x + 1)(x + 4)(x + 7)(x + 10) - 40.$
5. $(x - 3)(x - 2)(x - 6)(x - 4) - 12x^2.$
6. $(x - 8)(x + 4)(x + 2)(x - 4) + 9x^2.$
7. $x(2x + 1)(x - 2)(2x - 3) - 63.$
8. $x(3x + 2)(x - 2)(3x - 4) - 21.$
9. $(2x + 3)(2x + 7)(2x - 1)(2x - 5) + 175.$
10. $3x(3x - 1)(3x + 1)(3x + 2) - 3.$
11. $(x + 3)(x^2 - 1)(x + 5) - 9.$
12. $(x^2 - 4)(x + 1)(x + 5) - 45.$
13. $4(x + 1)(2x + 3)(x + 2)(2x + 5) - 360.$
14. $9(3x + 1)(x + 1)(3x - 4)(x - 2) + 13.$
15. Prove that the product of *any* four consecutive numbers increased by one is a perfect square.
16. Prove that the product of *any* four consecutive even numbers increased by 16 is a perfect square.
17. Prove that the product of *any* four consecutive odd numbers increased by 16 is a perfect square.

10. Type X. Factors of expressions which *can be reduced* to the form $a^2 - b^2$.

Example 1. Resolve into factors $4a^2 - b^2 - 9c^2 + 6bc$.

Since $6bc$ seems to be twice the product of two terms containing b and c , therefore terms containing b and c must be grouped together.

Thus the expression

$$\begin{aligned} &= 4a^2 - (b^2 + 9c^2 - 6bc) \\ &= (2a)^2 - (b - 3c)^2 \\ &= (2a + b - 3c)(2a - b + 3c). \end{aligned}$$

Example 2. Resolve into factors $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$.

The expression $= a^2 - b^2 - c^2 + d^2 - 2ad + 2bc$.

Since $2ad$ and $2bc$ appear to be twice the products of terms containing a, d and b, c respectively, therefore in this expression terms containing a, d and b, c are to be grouped within two pairs of brackets respectively.

Thus the expression

$$\begin{aligned} &= (a^2 - 2ad + d^2) - (b^2 - 2bc + c^2) \\ &= (a - d)^2 - (b - c)^2 \\ &= \{ (a - d) + (b - c) \} \{ (a - d) - (b - c) \} \\ &= (a + b - c - d)(a - b + c - d). \end{aligned}$$

EXERCISE 41.

Resolve into factors :

1. $x^2 + y^2 - z^2 + 2xy$.

2. $16x^2 - 4y^2 + 12yz - 9z^2$.

3. $4x^2 - 9y^2 - 25z^2 + 30yz$.

4. $16x^2 + 40xz - 36y^2 + 25z^2$.

5. $36z^2 - 60xz - 49y^2 + 25x^2$.

6. $2(ab + cd) - a^2 - b^2 + c^2 + d^2$.

7. $a^2 + 4b^2 - 9c^2 - 16d^2 - 2(2ab - 12cd)$.

8. $16a^2 - 9b^2 - 49c^2 + 25d^2 + 2(21bc + 20ad).$
9. $1 - 4a^2 + 9b^2 - 25c^2 + 20ac - 6b.$
10. $4a^2 - 9b^2 - 25c^2 + 30bc - 24a + 36.$
11. $25c^2 - 49b^2 - a^2 + 14ab - 20c + 4.$
12. $49a^2 - 1 + 16b^2 - 64c^2 + 16c - 56ab.$

Example 3. Resolve into factors $x^4 - 11x^2 + 1$.

$$\begin{aligned}
 x^4 - 11x^2 + 1 &= x^4 - 2x^2 + 1 - 9x^2. \\
 &= (x^2 - 1)^2 - (3x)^2. \\
 &= (x^2 - 1 + 3x)(x^2 - 1 - 3x) \\
 &= (x^2 + 3x - 1)(x^2 - 3x - 1).
 \end{aligned}$$

Example 4. Resolve into factors $4x^4 + 81$.

$$\begin{aligned}
 4x^4 + 81 &= (2x^2)^2 + (9)^2 + 2(2x^2)(9) - 2(2x^2)(9) \\
 &= (2x^2 + 9)^2 - (6x)^2 \\
 &= (2x^2 + 6x + 9)(2x^2 - 6x + 9).
 \end{aligned}$$

Example 5. Resolve into factors $(a^2 - 2ab) - (c^2 - 2bc)$.

$$\begin{aligned}
 (a^2 - 2ab) - (c^2 - 2bc) &= a^2 - 2ab - c^2 + 2bc \\
 &= a^2 - 2ab + b^2 - c^2 + 2bc - b^2 \\
 &= (a^2 - 2ab + b^2) - (c^2 - 2bc + b^2) \\
 &= (a - b)^2 - (c - b)^2 \\
 &= \{ (a - b) + (c - b) \} \{ (a - b) - (c - b) \} \\
 &= (a - b + c - b)(a - b - c + b) \\
 &= (a - 2b + c)(a - c).
 \end{aligned}$$

Resolve into factors :

- | | |
|--------------------------------|---------------------------------|
| 13. $a^4 + 2a^2 + 9.$ | 14. $a^4 + 7a^2 + 16.$ |
| 15. $4a^4 + 8a^2b^2 + 9b^4.$ | 16. $9a^4 - a^2b^2 + 16b^4.$ |
| 17. $4a^4 - 37a^2 + 9.$ | 18. $9a^4 - 33a^2b^2 + 16b^4.$ |
| 19. $25a^4 - 19a^2b^2 + 9b^4.$ | 20. $a^4 - 35a^2b^2 + 25b^4.$ |
| 21. $a^4 - 16a^2b^2 + 36b^4.$ | 22. $49a^4 - 60a^2b^2 + 16b^4.$ |
| 23. $a^4 + a^2b^2 + b^4.$ | 24. $625 + 4a^4.$ |
| 25. $a^4 + 4.$ | 26. $a^4 + 64.$ |
| 27. $9a^4 + 36b^4.$ | 28. $64p^4 + 81q^4.$ |

29. $a^4 + a^2 + 1$. 30. $a^8 + a^4 + 1$.
 31. $a^8 + a^4b^4 + b^8$. 32. $25a^4 + 35a^2b^2 + 36b^4$.
 33. $(9p^2 - 6pq) - (r^2 - 2qr)$. 34. $(a^2 - 6ab) - (25c^2 - 30bc)$.
 35. $(25a^2 - 20ab) - 3c(3c - 4b)$. 36. $7a(7a - 4b) - 3c(3c - 4b)$.

11. Type XI. Factorisation by suitable arrangement and grouping of terms.

Example 1. Resolve into factors :

$$9a^2 - 24ab + 16b^2 - 6a + 8b.$$

Grouping together terms of the same degree, we have the expression

$$\begin{aligned} &= (9a^2 - 24ab + 16b^2) - (6a - 8b) \\ &= (3a - 4b)^2 - 2(3a - 4b) \\ &= (3a - 4b)(3a - 4b - 2). \end{aligned}$$

Example 2. Resolve into factors

$$a^2 - b^2 - 9c^2 + 6bc + a + b - 3c.$$

Grouping together terms of the same degree, we have the expression

$$\begin{aligned} &= (a^2 - b^2 - 9c^2 + 6bc) + (a + b - 3c) \\ &= \{ (a^2) - (b - 3c)^2 \} + (a + b - 3c) \\ &= (a + b - 3c)(a - b + 3c) + (a + b - 3c) \\ &= (a + b - 3c)(a - b + 3c + 1). \end{aligned}$$

EXERCISE 42.

Resolve into factors :

1. $a^2 + 2ab + b^2 + a + b$.
2. $a^2 - 2ab + b^2 - a + b$.
3. $4a^2 + 12ab + 9b^2 - 8a - 12b$.
4. $a^2 - 10ab + 25b^2 - 3a + 15b$.
5. $a^2 - b^2 - c^2 + 2bc + a + b - c$.
6. $a^2 - b^2 - c^2 - 2bc + a - b - c$.
7. $4a^2 - b^2 - 9c^2 + 6bc - 6a + 3b - 9c$.
8. $9a^2 - 4c^2 - 24ab + 16b^2 - 15a + 20b + 10c$.

Example 3. Resolve into factors $a^4 + a^2c^2 - b^2c^2 - b^4$.

Combining similar terms, we have the expression

$$\begin{aligned} &= (a^4 - b^4) + (a^2c^2 - b^2c^2) \\ &= (a^2 - b^2)(a^2 + b^2) + c^2(a^2 - b^2) \\ &= (a^2 - b^2)(a^2 + b^2 + c^2) \\ &= (a - b)(a + b)(a^2 + b^2 + c^2). \end{aligned}$$

Example 4. Resolve into factors $a^3 + 5a^2 + 15a + 27$.

Combining 1st and 4th terms and 2nd and 3rd, we get

$$\begin{aligned} &(a^3 + 27) + (5a^2 + 15a) \\ &= \{ (a)^3 + (3)^3 \} + 5a(a + 3) \\ &= (a + 3)(a^2 - 3a + 9) + 5a(a + 3) \\ &= (a + 3)(a^2 - 3a + 9 + 5a) \\ &= (a + 3)(a^2 + 2a + 9). \end{aligned}$$

Example 5. Resolve into factors

$$a^3x^2 + x^5 - 2a^3bx + a^3b^2 + b^2x^3 - 2bx^4$$

Grouping together terms containing a^3 within one pair of brackets and the rest within another pair of brackets, we have the expression

$$\begin{aligned} &= (a^3x^2 - 2a^3bx + a^3b^2) + (x^5 + b^2x^3 - 2bx^4) \\ &= a^3(x^2 - 2bx + b^2) + x^3(x^2 - 2bx + b^2) \\ &= (x^2 - 2bx + b^2)(a^3 + x^3) \\ &= (x - b)^2(a + x)(a^2 - ax + x^2). \end{aligned}$$

Resolve into factors :

9. $a^2 + ab - bc - c^2$.
10. $4a^2 + 8ac - 12bc - 9b^2$.
11. $9a^2 - 21ac + 35bc - 25b^2$.
12. $a^3 + a^2 + a + 1$.
13. $a^3 + a^2 - a - 1$.
14. $a^3 - 2a^2b + 2ab^2 - b^3$.
15. $a^3 + 5a^2 - 10a - 8$.
16. $8a^3 + 18a^2b - 27ab^2 - 27b^3$.
17. $8a^3 + 14a^2b - 21ab^2 - 27b^3$.
18. $27a^3 - 15a^2b - 20ab^2 + 64b^3$.
19. $xy(a^2 + b^2) + ab(x^2 + y^2)$.
20. $a^2(a + 2b) + b^2(2a + b)$.
21. $a^4 - 5a^2c^2 + 20b^2c^2 - 16b^4$.
22. $16a^4 - 28a^2c^2 + 63b^2c^2 - 81b^4$.
23. $a^4 - 2a^3b + 2a^2b^2 - 2ab^3 + b^4$.

24. $a^4 - 5a^3b + 4a^2b^2 - 10ab^3 + 4b^4.$

25. $a^2p^2 - 3a^2q^2 - b^2p^2 + 2a^2pq - 2b^2pq + 3b^2q^2.$

12. Type XII. To factorise an expression, when one of its factors is known.

Example 1. Factorise $x^3 + 5x^2a + 2xa^2 - 8a^3$, when one of its factors is $x + 2a$.

Here we decompose the expression in such a way as to take out $x + 2a$ as a factor.

$$\begin{aligned} \text{The expression} &= x^2(x + 2a) + 3x^2a + 2xa^2 - 8a^3 \\ &= x^2(x + 2a) + 3xa(x + 2a) - 4xa^2 - 8a^3 \\ &= x^2(x + 2a) + 3xa(x + 2a) - 4a^2(x + 2a) \\ &= (x + 2a)(x^2 + 3xa - 4a^2) \\ &= (x + 2a)(x + 4a)(x - a). \end{aligned}$$

EXERCISE 43.

Factorise the following expressions, having given one factor of each :

	Expression.	Given factor.
1.	$2a^3 + 9a^2 - 8a - 15.$	$a + 5.$
2.	$4a^3 - 15a^2 + 3a + 18.$	$a - 3.$
3.	$2a^3 - 9a^2 - 27a + 54.$	$2a - 3.$
4.	$2a^3 - 7a^2 - 7a + 12.$	$2a + 3.$
5.	$a^3 + a^2b + ab^2 - 3b^3.$	$a - b.$
6.	$a^3 - 4a^2b + 5ab^2 - 2b^3.$	$a - 2b.$
7.	$a^3 + 3a^2b - 18ab^2 - 40b^3.$	$a + 2b.$
8.	$2a^3 - 3a^2b + 7ab^2 - 3b^3.$	$2a - b.$
9.	$6a^3 - 17a^2b + 6ab^2 + 8b^3.$	$3a - 4b.$
10.	$12a^3 + 5a^2b - 11ab^2 - 6b^3.$	$4a + 3b.$

The steps employed in the solution of example 1 can be abbreviated by practice, as illustrated in the next example.

Example 2. Factorise $2x^4 + x^3 - 8x^2 + 11x - 12$, when one of its factors is $2x - 3$.

The expression, when properly decomposed

$$\begin{aligned} &= 2x^4 - 3x^3 + 4x^3 - 6x^2 - 2x^2 + 3x + 8x - 12 \\ &= x^3(2x - 3) + 2x^2(2x - 3) - x(2x - 3) + 4(2x - 3) \\ &= (2x - 3)(x^3 + 2x^2 - x + 4). \end{aligned}$$

Factorise the following expressions, having given one factor of each :

	Expression.	Given factor.
11.	$2x^4 - x^3 - x^2 - x - 3.$	$2x - 3.$
12.	$2x^4 - x^3 - 4x^2 - x - 6.$	$2x + 3.$
13.	$x^4 - 6x^3 + 7x^2 + 5x - 4.$	$x - 4.$
14.	$2x^4 + 7x^3 - x^2 + 11x - 4.$	$x + 4.$
15.	$2x^4 + 5x^3 - x^2 - 5x + 2.$	$2x - 1.$
16.	$2x^4 + 3x^3 + 2x^2 - 1.$	$x^2 + x + 1.$
17.	$x^4 + x^3 + 2x^2 - x + 3.$	$x^2 - x + 1.$
18.	$2x^4 - 9x^3 + 11x^2 - 7x + 3.$	$2x^2 - x + 1.$
19.	$x^4 - 6x^3 + 13x^2 - 16x + 12.$	$x^2 - x + 2.$
20.	$x^4 - 4x^2 + 12x - 9.$	$x^2 - 2x + 3.$

EXERCISE 44.

[Revision]

Factorise :

1. $\frac{a^4}{81} - \frac{b^4}{16}.$

2. $1 - (a + b)^2.$

3. $a^3 - b^3 - a + b.$

4. $a^2 - 4b^2 - 4bc - c^2.$

5. $1 - \frac{x^3}{y^3}.$

6. $a^{16} + a^8 + 1.$

7. $a^{12} - 1.$

8. $a^3 + 7a^2 + 7a + 1.$

9. $x^2 - 2ax + (a - b)(a + b).$

10. $a^2 - 6ab + 9b^2 - a + 3b - 12.$
11. $(5a + 2)^4 - (3a - 1)^4.$
12. $(a^2 + 3a)^2 + 5(a^2 + 3a) - 84.$
13. $(a + b)^2 - 2(a^2 - b^2) + (a - b)^2.$
14. $(a^2 + ab + b^2)^2 - (a^2 - ab - b^2)^2.$
15. $49(a - b)^2 - 25(a + b)^2.$
16. $(a + b)^2 + 2c(a + b) + c^2 - d^2.$
17. $(ax + by)^2 - (ax - by)^2.$
18. $3a^3 - 6a^2 - 105a.$
19. $ax^2 + (a + b)xy + by^2.$
20. $a(a + c) - b(b + c).$
21. $(ax - by)^2 + (bx + ay)^2.$
22. $6a^3 + 4a^2b + 9ab^2 + 6b^3.$
23. $a^2b^2 - a^2 - b^2 + 1.$
24. $a^2b^2c^2 - a^2c - b^2c + 1.$
25. $3 - 12(a - b)^2.$
26. $1 - 12(x - y) + (6x - 6y)^2.$
27. $x^3 - \frac{4}{x}.$
28. $x^3 + 64.$
29. $x^4 + 64.$
30. $x^8 - y^8.$
31. $x^9 - y^9.$
32. $a^{3m} - b^{3n}.$
33. $27a^3b - 48ab^3.$
34. $\frac{x^3}{27} - \frac{y^3}{64}.$
35. $9x^2 - 41xy + 20y^2.$
36. $9x^2 + 41xy - 20y^2.$
37. $1 - a + b - 90(a - b)^2.$
38. $1 + a + b + c + ab + ac + bc + abc.$
39. $x^3 - .001.$
40. $12x^4 + 243y^4.$
41. $21(1 - x^2) + 40x.$
42. $9x^2 - 20 - (x - 1)^2.$
43. $2 - 3x - 2x^2 + (2x - 1)(3x - 4).$
44. $(x + y)^2 - x^2 + y^2 - 2(x - y)^2.$
45. $(x + y)^4 + (x^2 - y^2)^2 + (x - y)^4.$
46. $2(p + q)^4 + 5(p^2 - q^2)^2 + 2(p - q)^4.$
47. $ab(x + y)^2 - (a + b)(x^2 - y^2) + (x - y)^2.$
48. $(a^2 - b^2) + \left(\frac{1}{a^2} - \frac{1}{b^2}\right).$
49. $ax^2 + (a^2 + 1)x + a.$
50. $(2a - 3b)^2 - (2a - 3b)(a + 2b) - 20(a + 2b)^2.$
51. $(a^2 - 5a)^2 - 3(a^2 - 5a) - 28.$
52. $(2x^2 - 3xy)^2 - 5(2x^2 - 3xy)(2xy - y^2) - 24(2xy - y^2)^2.$
53. $x^3 - 6x^2y + 18xy^2 - 27y^3.$
54. $\frac{a^3}{8} + b^3 + \frac{c^3}{27} - \frac{abc}{2}.$

FACTORS

55. $x^2 + \left(p - \frac{1}{p}\right)xy - y^2$. 56. $4x^4 + (7a)^4$.
57. $x^4 - 22x^2 + 9$. 58. $(a + b + c)^2 - a^2 + b^2 - c^2$.
59. $(2a + b + c)^2 - a^2 - (a + b + c)^2$.
60. $(a + b)^2 + (b + c)^2 - (c + d)^2 - (d + a)^2$.
61. $(a + b)^3 + (b + c)^3 + (c + d)^3 + (d + a)^3$.
62. $4(ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$.
63. $(s - a)^3 + (s - b)^3 + (s - c)^3 - 3(s - a)(s - b)(s - c)$.
64. $(x + 2)(x + 3)(x + 4)(x + 5) - 360$.
65. $(a - 3)(a - 2)(a + 6)(a + 9) + 3a^2$.
66. $(3n + 1)(3n + 4)(3n + 7)(3n + 10) + 81$.
67. $a^2(a + b - c)^2 - c^2(b + c - a)^2$.
68. $11x^3y^3 - 61x^2y^2 - 30xy$.
69. $(a^2 - b^2)x^2 - 2bx - 1$.
70. $(a^2 - b^2)(c^2 - d^2) + 4abcd$.
71. $(a^2 - b^2)(c^2 + d^2) + 2(a^2 + b^2)cd$.
72. $x^4(x^4 - 1) - a^4(a^4 - 1)$.
73. $x^2(x^4 - 1) + y^2(y^4 - 1)$.
74. $y^2(y - 1) - x^2(x - 1)$.
75. $1 + x - x^2 + x^4$. 76. $z(x + y)^2 - y(z + x)^2$.
77. $a(a - 2) + b(b - 2) + 2ab$. 78. $(x^3 - y^3) + 2(x^4 - y^4)$.
79. $(25p^2 - 10pq) - (r^2 - 2rq)$.
80. $(3x + 1)(x + 2)^2 - (x + 2)(x - 1)^2$.
81. $(x + 1)(x + 2)(x + 3) - 2x - 2$.
82. $x(x - 2y)^3 - y(y - 2x)^3$.
83. $a^2px + a^2rz - b^2qy + b^2rz - a^2qy + b^2px$.
84. $x - 3$ is one factor of $x^3 - 7x^2 - 9x - 63$; find the others.
85. $2x - 3$ is one factor of $2x^3 - 11x^2 + 22x - 15$; find the others.
86. Prove that $(x^2 + x - 2)(x^2 - 4x + 3)(x^2 - x - 6)$ is a perfect square.

CHAPTER VI

APPLICATION OF FORMULÆ AND FACTORS

By a judicious use of factors and formulæ the solution of many questions in various rules of Algebra can be simplified as illustrated below.

First Four Rules

Example 1. Subtract $(x^2 - 7x + 15)^2$ from $(x^2 - 7x - 15)^2$.

$$\begin{aligned} \text{The result} &= (x^2 - 7x - 15)^2 - (x^2 - 7x + 15)^2 \\ &= [(x^2 - 7x - 15) + (x^2 - 7x + 15)][(x^2 - 7x - 15) \\ &\quad - (x^2 - 7x + 15)] \\ &= (x^2 - 7x - 15 + x^2 - 7x + 15) \\ &\quad (x^2 - 7x - 15 - x^2 + 7x - 15) \\ &= (2x^2 - 14x)(-30) = -60x(x - 7). \end{aligned}$$

EXERCISE 45.

Subtract :

1. $(x^2 - 5x + 12)^2$ from $(x^2 - 5x - 12)^2$.
2. $(x^2 - 3x + 8)^2$ from $(x^2 + 3x + 8)^2$.
3. $x^2(z + x - y)^2$ from $y^2(y + z - x)^2$.
4. $(5x^2 - 5xy - 11y^2)^2$ from $(6x^2 - 5xy + 11y^2)^2$.

Example 2. Find the continued product of

$$(a - b), (a + b), (a^2 - ab + b^2) \text{ and } (a^2 + ab + b^2).$$

$$\begin{aligned} \text{The result} &= (a - b)(a^2 + ab + b^2) \{ (a + b)(a^2 - ab + b^2) \} \\ &= (a^3 - b^3)(a^3 + b^3) = a^6 - b^6. \end{aligned}$$

5. Multiply $1 - a^2 + a^4$ by $1 + a^2 + a^4$.
6. Multiply $(a + b)^2 - (a + b) + 1$ by $(a + b)^2 + (a + b) + 1$.

Find the continued product of :

7. $(a - b)^2, (a + b)^2$ and $(a^2 + b^2)^2$.
8. $a^2 - ab + b^2, a^2 + ab + b^2$ and $a^4 - a^2b^2 + b^4$.
9. $1 + a(1 + a), 1 - a(1 + a), 1 + a^2(1 + a)^2$.
10. $(a + b + c), (a + b - c), (b + c - a), (c + a - b)$.
11. $(x + y + z), (x - y - z), (y - z - x), (z - x - y)$.

12. $(a+b+c-d), (a+b-c+d), (a-b+c+d), (-a+b+c+d)$.
 13. $a^2+2ab-3b^2, a^2+3ab+2b^2, a^2-5ab+6b^2$.
 14. $2a^2+a-3, 2a^2-a-3, a^4+a^2+1$.
 15. $a^2-2ab+4b^2, a^2+2ab+4b^2, a^4-4a^2b^2+16b^4$.
 16. $(x+3)^2, (x^2+9)^2, (x^2-6x+9)$.
 17. $(x^2+4x+4), (x^2-4x+4), (x^4+8x^2+16)$.

Example 3. Divide $64a^3-27b^3-8c^3-72abc$ by $4a-3b-2c$.

$$\begin{aligned}\text{The dividend} &= (4a)^3 + (-3b)^3 + (-2c)^3 - 3(4a)(-3b)(-2c) \\ &= (4a-3b-2c) \times \{ (4a)^2 + (-3b)^2 + (-2c)^2 - (4a)(-3b) \\ &\quad - (4a)(-2c) - (-3b)(-2c) \} \\ &= (4a-3b-2c)(16a^2+9b^2+4c^2+12ab+8ac-6bc).\end{aligned}$$

$$\therefore \text{the quotient} = 16a^2+9b^2+4c^2+12ab+8ac-6bc.$$

Divide :

18. $(a+b)^2-(b-c)^2$ by $a+c$.
 19. $a^8+a^4b^4+b^8$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.
 20. $a^8-b^8+a^2b^2(a^4-b^4)$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.
 21. $(a^2-3a+2)(a-3)$ by a^2-5a+6 .
 22. $(a^2-3a+2)(a+4)$ by a^2+3a-4 .
 23. $(a^2-5ab+6b^2)(a-4b)$ by $a^2-7ab+12b^2$.
 24. $(a^2+ab+b^2)(a^3+b^3)$ by $a^4+a^2b^2+b^4$.
 25. a^6-b^6 by $a^3-2a^2b+2ab^2-b^3$.
 26. $(a-b)^3+c^3$ by $a-b+c$.
 27. $(a+b)^3-8c^3$ by $a+b-2c$.
 28. $a^3+4a-39$ by $a-3$.
 29. $a^3-4a^2b+24b^3$ by $a+2b$.
 30. a^6-26a^3-27 by $(a^2+3a+9)(a^2-a+1)$.
 31. $27a^3-b^3-8c^3-18abc$ by $3a-b-2c$.
 32. $125x^3-216y^3+z^3+90xyz$ by $5x-6y+z$.
 33. $(a^6+b^6-2a^3b^3)$ by $(a-b)^2$.
 34. $(2x^2-6y^2-4z^2+11yz+2zx+xy)$ by $(x+2y-z)$.
 35. $(a^2-b^2)^2+4ab-1$ by $(a+b)^2-1$.
 36. Divide the product of $2x^2+11x-21$ and $3x^2-20x-7$ by x^2-49 .

37. The product of two expressions is $(4y - 3x)^3 - (4x - 3y)^3$ and one of them is $7(y - x)$; find the other.

Shew that

38. $(3x^2 - 5x + 4)^2 - (2x^2 + x - 4)^2$ is exactly divisible by $(x - 4)(x - 2)$.

39. $(ax + by)^2 - (bx + ay)^2$ is exactly divisible by $(a + b)(x + y)$.

40. $(x - 4y)^3 - (y - 4x)^3 + (2y - 3x)^3 - (2x - 3y)^3$ is exactly divisible by $5(x - y)$.

41. $(ax + by + cz)^3 + (bx + cy + az)^3$ is exactly divisible by $(a + b)x + (b + c)y + (c + a)z$.

Numerical Fractions and Evaluation

Example 4. Simplify by factors $\frac{(1.53)^2 - (.47)^2}{1.06}$.

$$\begin{aligned}\frac{(1.53)^2 - (.47)^2}{1.06} &= \frac{(1.53 + .47)(1.53 - .47)}{1.53 - .47} \\ &= 1.53 + .47 = 2.\end{aligned}$$

Simplify by factors :

42. $\frac{.345 \times .345 - .26 \times .26}{.085}$ 43. $\frac{.72 \times .72 - .48 \times .48}{1.2}$

44. $\frac{(137\frac{3}{4})^2 - (128\frac{1}{4})^2}{9\frac{1}{2}}$ 45. $\frac{(125\frac{5}{9})^2 - (12\frac{4}{9})^2}{138}$ 46. $\frac{(73)^3 - (57)^3}{73 - 57}$

47. $\frac{(.43)^3 + (.35)^3}{.78}$ 48. $\frac{.841 \times .841 \times .841 + .159 \times .159 \times .159}{.841 \times .841 - .841 \times .159 + .159 \times .159}$

49. $\frac{11.42 \times 11.42 \times 11.42 - 1.42 \times 1.42 \times 1.42}{11.42 \times 11.42 + 11.42 \times 1.42 + 1.42 \times 1.42}$

50. $\frac{62 \times 62 - 38 \times 38}{62 \times 62 + 2 \times 62 \times 38 + 38 \times 38}$ 51. $\frac{(2.35)^3 - (1.65)^3}{(2.35)^2 - (1.65)^2}$

52. $\frac{.73 \times .78 \times .73 + .59 \times .59 \times .59}{.073 \times .073 - .073 \times .059 + .059 \times .059}$

Example 5. Find the value of $x^3 + y^3 + z^3 - 3xyz$ when $x = 329$, $y = 334$ and $z = 337$.

$$\begin{aligned}
\text{The expression} &= \frac{1}{2}(x+y+z) \{ (x-y)^2 + (y-z)^2 + (z-x)^2 \} \\
&= \frac{1}{2}(329+334+337) \{ (329-334)^2 + (334-337)^2 \\
&\quad + (337-329)^2 \} \\
&= \frac{1}{2} \times 1000(25+9+64) \\
&= 500 \times 98 = 49000.
\end{aligned}$$

Example 6. Express $x^4 + x^2y^2 + y^4$ in terms of a and b when $x+y=2a$ and $x-y=2b$.

From the given relations, we have $x=a+b$, $y=a-b$ and $xy=a^2-b^2$.

$$\begin{aligned}
\text{Now } x^4 + x^2y^2 + y^4 &= (x^2 + xy + y^2)(x^2 - xy + y^2) \\
&= \{ (x+y)^2 - xy \} \{ (x-y)^2 + xy \} \\
&= \{ (2a)^2 - (a^2 - b^2) \} \{ (2b)^2 + (a^2 - b^2) \} \\
&= (3a^2 + b^2)(3b^2 + a^2).
\end{aligned}$$

Find the value of :

53. $a^3 + b^3 - c^3 + 3abc$ when $a = \cdot 457$, $b = \cdot 236$, $c = \cdot 693$.

54. $a^3 + b^3 + c^3 - 3abc$ when $a = 357$, $b = 367$, $c = 377$.

55. $x^3 + 8y^3 - 27 + 18xy$ when $x + 2y = 3$.

56. $8x^3 + y^3 - 1 + 6xy$ when $x = \cdot 357$, $y = \cdot 286$.

57. $a^3 + b^3 + c^3 - 3abc$ when $a = \cdot 326$, $b = \cdot 336$, $c = \cdot 338$.

Express :

58. $a^3 + b^3$ in terms of x and y if $x = a + b$, $y = a - b$.

59. $x^4 - 6x^2y^2 + y^4$ in terms of a and b if $x + y = 2a$,
 $x - y = 2b$.

60. Find the value of $x^2 - y^2 + 4x + 14y - 45$ when $x + y = 23$ and $x - y = 7$. [*Hint.* The expression $= (x+2)^2 - (y-7)^2$.]

61. If $x = b + c - 2a$, $y = c + a - 2b$, $z = a + b - 2c$, find the value of $y^2 + z^2 - x^2 + 2yz$.

H. C. F. by Factors

Example 7. Find the highest common factor of :

$$a^3 + 2a^2b + ab^2, 6a^3 + 4a^2b - 2ab^2, 3(a^2 + ab)^2.$$

We have $a^3 + 2a^2b + ab^2 = a(a^2 + 2ab + b^2) = a(a+b)^2 \dots$ (i)

$$6a^3 + 4a^2b - 2ab^2 = 2a(3a^2 + 2ab - b^2)$$

$$= 2a(a+b)(3a-b) \dots$$
 (ii)

$$3(a^2 + ab)^2 = 3a^2(a + b)^2 \quad \dots \text{(iii)}$$

\therefore From (i), (ii) and (iii) the H. C. F. is obviously
 $= a(a + b).$

Find by factors the H. C. F. of :

- | | |
|---|------------------------------------|
| 62. $a^2 - b^2, a(a - b).$ | 63. $a^3 - 3a^2b, 3ab^2 - 9b^3.$ |
| 64. $a^4 - 4a^2b^2, ab^2 + 2b^3.$ | 65. $a^3 + 8, a^2 + 5a + 6.$ |
| 66. $a^3 - 27, (a - 3)^3.$ | 67. $a^2 - a, (a - 1)^2, a^3 - 1.$ |
| 68. $a^2 - 3a - 18, a^2 + 5a + 6.$ | |
| 69. $9x^4y^4 - 36x^2y^6, 24x^4y^4 - 48x^3y^3.$ | |
| 70. $48a^2x^2(a + x)^2(a^2x^2 - a^2x), 64(a^2x^5 - a^5x^2)(ax^3 + a^2x^2).$ | |
| 71. $4a^2b(x^2 - 1), 6ab^2(x + 1)^2, 8ab(x^2 - 2x + 1).$ | |
| 72. $a^5b^2 - a^4b^3 + a^3b^4, 15a^3b + 30a^2b + 15ab.$ | |
| 73. $a^2 + 5a + 6, a^2 + 9a + 14, a^2 - 7a - 18.$ | |
| 74. $a^2 - c^2 + b^2 - 2ab, a^2 - b^2 - c^2 - 2bc.$ | |
| 75. $x^2 + 7x + 12, x^2 + 9x + 20, x^3 + 64.$ | |
| 76. $a^2 - 6ab + 8b^2, a^2 - 8ab + 16b^2, a^3 - 64b^3.$ | |
| 77. $12a^2 - a - 20, 15a^2 - 38a + 24, 21a^2 - 52a + 32.$ | |
| 78. $x^3 - y^3 - z^3 - 3xyz, x^2 - y^2 - z^2 - 2yz.$ | |
| 79. $21a^4 - 8a^3 - 45a^2, 42a^5 + 26a^4 - 36a^3.$ | |

Sometimes one of the expressions can be easily resolved into factors. Then we find by trial which of these factors is common to both.

Example 8. Find the H. C. F. of $5x^2 - 3x - 8$

and $x^4 - 2x^3 - 4x - 7.$

$$\begin{aligned} 5x^2 - 3x - 8 &= 5x^2 - 8x + 5x - 8 \\ &= x(5x - 8) + (5x - 8) \\ &= (5x - 8)(x + 1). \end{aligned}$$

Obviously, $5x - 8$ cannot be a factor of the 2nd expression as it begins with $5x$ and ends with 8 .

Thus $x + 1$ may be a factor, of the 2nd expression. Arranging the terms of the 2nd expression so as to take out, if possible, $x + 1$ as a factor, we have the expression

$$\begin{aligned} &= x^4 + x^3 - 3x^3 - 3x^2 + 3x^2 + 3x - 7x - 7 \\ &= x^3(x + 1) - 3x^2(x + 1) + 3x(x + 1) - 7(x + 1) \end{aligned}$$

$$= (x+1)(x^3 - 3x^2 + 3x - 7).$$

$\therefore x+1$ is the H. C. F.

Find by factors the H. C. F. of:

80. $2a^2 - 7a + 6, a^3 - 5a^2 + a + 10.$

81. $3a^2 + a - 14, a^3 + 2a^2 - 3a - 10.$

82. $a^2 - 4a - 21, a^3 + 3a^2 - 3a - 9.$

83. $a^4 - 1, 3a^5 + 2a^4 + 4a^3 + 2a^2 + a.$

84. $a^2 + a - 6, a^2 + 3a - 10, a^3 + a^2 - 5a - 2.$

85. $a^2 + a - 12, a^2 - 2a - 3, a^3 - 4a^2 - 2a + 15.$

86. $8x^3 - 1, 10x^3 - 19x^2 + 5x + 1.$

87. $x^3 - y^3 - 1 - 3xy, x^2 - 4xy + 3y^2 + 2x - 3.$

L. C. M. by Factors

Example 9. Find by factors the L. C. M. of:

$$(a^2 + 2a)^2, 2a^4 + 3a^3 - 2a^2 \text{ and } 2a^3 - 3a^2 - 14a.$$

$$(a^2 + 2a)^2 = \{ a(a+2) \}^2 = a^2(a+2)^2,$$

$$2a^4 + 3a^3 - 2a^2 = a^2(2a^2 + 3a - 2) = a^2(a+2)(2a-1).$$

$$2a^3 - 3a^2 - 14a = a(2a^2 - 3a - 14) = a(a+2)(2a-7).$$

$$\therefore \text{the L. C. M.} = a^2(a+2)^2(2a-1)(2a-7).$$

Find by factors the L. C. M. of:

88. $(a-1)^2, (a^3-a).$ 89. $a^2-4, a^3+8.$

90. $(a-1)^3, a^3-1.$ 91. $a^2+4a+3, a^2+5a+6.$

92. $a^2-ab-2b^2, a^2-5ab+6b^2, a^2-2ab-3b^2.$

93. $x^2+5ax+4a^2, x^2+11ax+28a^2, x^2+20ax+91a^2.$

94. $x^2+3x+2, x^2+4x+3, x^2+5x+6.$

95. $a^2-1, a^3+1, a^3-1.$

96. $a^2-1, a^2+1, a^4+1, a^8-1.$

97. $a^2-1, a^3+1, a^3-1, a^6+1.$

98. $a^2-4b^2, (a+2b)^2, (a-2b)^2.$

99. $6b^2(a-b)^2, 9a^3(a-b)^3, 12ab(a-b)^5.$

100. $18(a+b)^2(a^3-b^3), 24(a-b)^2(a+b)^3, 36(a^2-b^2)^2.$

101. $6a^2b^2(a^3-b^3), 15b^4(a-b)^3, 12a^3b(a-b)(a^2-b^2).$

102. $12(a-b)^2(a^2+b^2), 9(a-b)^3(a+b), 6(a^4-b^4),$

$$18(a+b)^3(a-b).$$

103. $a^4 + a^2b^2 + b^4, a^3b + b^4, (a^2 - ab)^3.$

104. $x^3 - 3x^2 + 3x - 1$, $x^3 - x^2 - x + 1$, $x^4 - 2x^3 + 2x - 1$,
 $x^4 - 2x^3 + 2x^2 - 2x + 1$.
105. $3a^3 - 18a^2b + 27ab^2$, $4a^4 + 24a^3b + 36a^2b^2$, $6a^4 - 54a^2b^2$.
106. $8a^3 + 27b^3$, $8a^3 - 27b^3$, $16a^4 + 36a^2b^2 + 81b^4$.
107. $6a^2 - a - 1$, $3a^2 + 7a + 2$, $2a^2 + 3a - 2$.
108. $2a^2 + 11a - 21$, $3a^2 + 25a + 28$, $a^4 + 5a^3 - 14a^2 - 5a - 35$.
109. $x^3 - a^3$, $x^2 - ax + a^2$, $x^3 + a^3$.
110. $x^2 + (b - c)x - bc$, $x^2 + (c - a)x - ca$, $x^2 + (a - b)x - ab$.
111. $1 + 4a + 4a^2 - 16a^4$, $1 + 2a - 8a^3 - 16a^4$.

Simplification of Fractions by Factors

Example 10 Reduce to the lowest terms $\frac{2x^2 + 3xy - 2y^2}{2x^2 - 7xy + 3y^2}$.

$$\begin{aligned} \frac{2x^2 + 3xy - 2y^2}{2x^2 - 7xy + 3y^2} &= \frac{2x^2 + 4xy - xy - 2y^2}{2x^2 - 6xy - xy + 3y^2} \\ &= \frac{2x(x + 2y) - y(x + 2y)}{2x(x - 3y) - y(x - 3y)} \\ &= \frac{(x + 2y)(2x - y)}{(x - 3y)(2x - y)} = \frac{x + 2y}{x - 3y}. \end{aligned}$$

Example 11: Simplify by factors

$$\frac{x^3 - 1}{x^2 + x - 6} \times \frac{x^2 - 4x + 4}{x^2 - 4x + 3} \times \frac{x^2 - 9}{x^4 + x^2 + 1}.$$

The expression

$$\begin{aligned} &= \frac{(x - 1)(x^2 + x + 1)}{(x + 3)(x - 2)} \times \frac{(x - 2)^2}{(x - 3)(x - 1)} \times \frac{(x + 3)(x - 3)}{(x^2 + x + 1)(x^2 - x + 1)} \\ &= \frac{x - 2}{x^2 - x + 1}. \end{aligned}$$

Reduce to the lowest terms ;

112. $\frac{x^2 + 3x + 2}{x^2 + 6x + 5}$ 113. $\frac{4(a + b)^2}{5(a^2 - b^2)}$ 114. $\frac{x^2 - 16x - 17}{x^2 - 22x + 85}$
115. $\frac{x^2 + 2x - 15}{x^2 + 9x + 20}$ 116. $\frac{1 - 7x + 12x^2}{1 - 8x + 15x^2}$ 117. $\frac{1 - 9x^2 + 14x^4}{1 - 4x^2 - 21x^4}$

$$118. \frac{6x^2 - 49x + 65}{14x^2 - 93x + 13}$$

$$120. \frac{x^2 - (a-b)x - ab}{x^3 + bx^2 + ax + ab}$$

$$122. \frac{a^2 - b^2 - 2bc - c^2}{a^2 + 2ab + b^2 - c^2}$$

$$124. \frac{4a^4 - 5a^2 + 1}{4a^4 - 6a^3 + 2a^2}$$

$$126. \frac{x-1}{x^3-1}$$

$$128. \frac{x^4-1}{x^6-1}$$

$$130. \frac{(x+1)^3 - (x-1)^3}{(x+1)^4 - (x-1)^4}$$

$$119. \frac{3abc - 3b^2c}{a - bc - b(1-c)}$$

$$121. \frac{2x^2 + xy - y^2}{x^3 + x^2y - x - y}$$

$$123. \frac{x^3 - x^2 - x + 1}{x^3 + x^2 + x + 1}$$

$$125. \frac{(x+a)^2 - (b+c)^2}{(x+b)^2 - (a+c)^2}$$

$$127. \frac{x^3-1}{x^4-1}$$

$$129. \frac{x^8-1}{x^{12}-1}$$

$$131. \frac{(x^2+6x+11)^2 - 9(x+3)^2}{(x^2+7x+9)^2 - (2x+5)^2}$$

Simplify by factors:

$$132. \frac{ac+bc}{a^2+ab+b^2} \times \frac{a^3-b^3}{a^2-b^2}$$

$$133. \frac{x(a^2-ab+b^2)}{a^2y-b^2y} \times \frac{(a^3+b^3)}{a^4+a^2b^2+b^4}$$

$$134. \frac{x^2-3x+2}{x^2+x-12} \times \frac{x^2+6x+8}{x^2-4}$$

$$135. \frac{x^2-4x-12}{x^2+2x-35} \times \frac{x^2+5x-14}{x^2-8x+12}$$

$$136. \frac{ax+by}{a^2x^2-b^2y^2} \times \frac{a^3x^3+b^3y^3}{a^4x^4+a^2b^2x^2y^2+b^4y^4}$$

$$137. \frac{x^2-1}{x^2+x-2} \times \frac{x^3+8}{x^4+4x^2+16} \div \frac{x^2+x}{x^3+2x^2+4x}$$

$$138. \frac{6x^2-5xy-6y^2}{4x^2-11xy+6y^2} \times \frac{5x^2-7xy-6y^2}{6x^2-11xy+3y^2} \times \frac{12x^2-13xy+3y^2}{15x^2+19xy+6y^2}$$

$$139. \frac{x^2-ax-2a^2}{x^2+ax-5cx-5ac} \times \frac{x^2+2bx-5cx-10bc}{x^2-2ax+2cx-4ac} \times \frac{x+2c}{x+2b}$$

$$140. \frac{x^3-y^3}{(x+y)^2} \times \frac{(x^3+y^3)}{x^2+xy-2y^2} \div \frac{x^4+x^2y^2+y^4}{x^2+3xy+2y^2}$$

$$141. \frac{x^3-a^3}{x^2-ax+bx-ab} \times \frac{x^2-b^2}{x^2+ax-bx-ab} \div \frac{x^2+ax+a^2}{x^2+ax+bx+ab}$$

$$142. \frac{a^2-b^2+c^2-2ac}{a^2+b^2-c^2-2ab} \times \frac{c^2-a^2-b^2+2ab}{b^2-c^2-a^2+2ac}$$

SECTIONAL REVISION II

TEST PAPERS

PAPER I

1. (i) State $(a \pm b)^2 = a^2 \pm 2ab + b^2$ in words and apply it in expanding $(5x \pm 4y)^2$.

(ii) Simplify $(3a - b)^2 + (a + 3b)^2$.

2. Find the value of $x^2 + \frac{1}{x^2}$ when $x + \frac{1}{x} = 6$.

3. Prove that (i) the difference between two numbers which are formed by the same two digits is exactly divisible by 9; (ii) the difference of their squares is exactly divisible by 99.

4. Factorise the following:

(i) $(2a - 1)^2 - (a - 2)^2$; (ii) $1 - 4a^2 - b^2 - 4ab$;
(iii) $24x^2 - 4x - 48$; (iv) $(x - 1)(x - 5)(x - 9)$
 $+ (x - 3)(x - 4)(x - 5)$.

5. Find the L. C. M. of $6a^2 - 7a + 2$, $3a^2 - 11a + 6$,
 $2a^2 - 7a + 3$.

6. Simplify $\frac{3x^2 - xy - 2y^2}{6x^2 - 5xy - 6y^2} \div \frac{x^2 + 3xy - 4y^2}{2x^2 + 5xy - 12y^2}$.

PAPER 2

1. (i) Find the missing term in the perfect squares:

(a) $81x^2 - 126xy + (\quad)$, (b) $36x^2 + (\quad) + 121y^2$.

(ii) Find by $(a \pm b)^2 = a^2 \pm 2ab + b^2$, the value of

(a) $(504)^2$, (b) $(495)^2$.

2. Find the value of $x^2 + \frac{1}{x^2}$ when $x - \frac{1}{x} = 8$.

3. (i) A class is required to work out examples 2, 5, 8, 11, 14, 17, 20, etc. in a set of examples. Find a formula which gives these numbers.

(ii) Write down the quotient of $x^5 - y^5$ by $x - y$.

4. (i) Shew that $(4x^2 - 5x + 7)^2 - (5x^2 + 14x + 2)^2$ is divisible by $x^2 + x + 1$ and find the quotient.

(ii) Prove the identity

$$(x-1)(2x+3) + (2x+1)(x-5) \equiv 3x+2)(x-3) + (x-2)(x+1).$$

5. Factorise (i) $x^3 + x^2 - 42x$.

(ii) $(a+b)^2 - 2(a^2 - b^2) - 15(a-b)^2$.

6. Find the L. C. M. of

$$35a^2 - 11a - 6 \text{ and } 40a^2 - 29a + 3.$$

PAPER 3

1. State $(a+b)(a-b) = a^2 - b^2$ in words; apply it in finding the product of $(p-2q+3r)(p-2q-3r)$ and the value of $(328)^2 - (323)^2$.

2. Find the value of $x^4 + \frac{1}{x^4}$ when $x - \frac{1}{x} = 4$.

3. Prove the identity $(n+3)^2 \equiv 3(n+2)^2 - 3(n+1)^2 + n^2$ and use it to express the value of d^2 in terms of a^2, b^2, c^2 where a, b, c, d are four consecutive numbers, of which a is the least.

4. Shew that $(4x^2 - 8x - 1)^2 - (2x^2 - 5x + 7)^2$ is divisible by $2x^2 - 3x - 8$, and express the quotient as the product of two factors.

5. Simplify the following:

(i) $\frac{(2a-9)^2 - (a-6)^2}{(a-5)^2}$.

(ii) $\frac{3(a-8)^3(a-9)^2 - (a-8)^2(a-9)^3}{2a^2 - 31a + 120}$.

6. Find the L. C. M. of $a^2 - 3ab - 10b^2, a^2 - 8ab + 15b^2$ and $a^2 + 2ab - 35b^2$.

PAPER 4

1. (i) State in words the formula

$$(x+a)(x+b)(x+c) = x^3 + (a+b+c)x^2 + (ab+ac+bc)x + abc,$$

and complete the following:

$$(p+2)(\quad -3)(p+\quad) \equiv p^3 + (\quad)p^2 + (\quad)p - 24.$$

(ii) Write down the product of $(x+3), (x-4), (x+5)$.

2. Find the value of $x^3 + \frac{1}{x^3}$ when $x + \frac{1}{x} = 4$.
3. If $a + b + c = 0$, shew that
 (i) $(a+b)(b+c)(c+a) = -abc$,
 and (ii) $a^3 + b^3 + c^3 = 3abc$.
4. If $a = 1$, $b = 2$, $c = 3$, find the value of
 (i) $\frac{(a+b+c)^3}{a^3 + b^3 + c^3}$, (ii) $a^{bc} + b^{ca} + c^{ab}$.
 (iii) $(bc)^a + (ca)^b + (ab)^c$.
5. Factorise (i) $16x^8 - 1$, (ii) $27a^5b^2 - 8a^2b^5$,
 (iii) $4(2a+3b)^2 - 9(a-b)^2$,
 (iv) $(2x^2 - 5x + 3)(2x^2 - 5x + 4) - 2$.
6. Simplify $\frac{3x^2 - 6xy}{4xy - 12y^2} \div \frac{4x^2 - 6xy}{6xy - 9y^2} \div \frac{4x^2 - 13xy + 3y^2}{4x^2 - 9xy + 2y^2}$.

PAPER 5

1. State in words $(a \pm b)^3 = a^3 \pm 3a^2b + 3ab^2 \pm b^3$
 $= a^3 \pm b^3 \pm 3ab(a \pm b)$,
 and fill in the blanks in the following
 $(5m - \quad)^3 \equiv (\quad) - 3(\quad)(\quad) + 3(5m)(\quad) - (2n)^3$.
2. Find the value of :
 (i) $a^3 + b^3$ if $a + b = 11$ and $ab = 30$,
 (ii) $a^3 - b^3$ if $a - b = 3$ and $ab = 40$,
 (iii) $8m^3 - 36m^2 + 54m - 50$ when $m = 3$
3. (i) Shew that $x^2 + y^2$ is less than $(x + y)^2$
 (ii) Find v in terms of u and k from $\frac{1}{u} - \frac{1}{v} = \frac{1}{k}$.
4. Factorise (i) $(x-3)(x+1)^3 + (x-1)^4 - 2(x-1)^3$,
 (ii) $x^4 + x^3 - 30x^2$,
 (iii) $x^2 - xz + yz - y^2$.
5. (i) Make n the subject of the formula $I = \frac{E}{\frac{r}{n} + R}$.
 (ii) Find the co-efficient of x^3 in the expansion of
 $(6x^3 - 5x^2 - 4x + 2)(3x^3 + 2x^2 + 5x - 1)$.

6. (i) Find the value of $a^2 + b^2 + c^2$ when $a + b + c = 15$ and $ab + ac + bc = 85$.

(ii) Simplify $\frac{x^2 + xy + y^2}{x^2 + 2xy + y^2} \times \frac{x^2 - xy + y^2}{x^2 - 2xy + y^2} \div \frac{x^6 - y^6}{(x^2 - y^2)^3}$.

PAPER 6

1. State in words $(a + b)(a^2 + ab + b^2) = a^3 + b^3$; apply it in writing down the product of

$(3 + 4a)(9 - 12a + 16a^2)$ and simplify

$(p^2 + 5)(p^4 - 5p^2 + 25) - (p^2 - 2)(p^4 + 2p^2 + 4)$.

2. Find the value of $x^3 - \frac{1}{x^3}$ when $x - \frac{1}{x} = p$.

3. Factorise (i) $(2x + y)^2 + 6y(2x + y) + 9y^2$,

(ii) $x^2(x + a) - x(x + a)(a + b) + ab(x + a)$.

4. (i) Find k when $(x - a)(x - 3a)(x + a)(x + 3a) + k$ is a perfect square.

(ii) Substitute $(m - 1)$ for a in the expression $2a^3 - 3a^2 + 4a - 5$ and arrange the result in descending powers of m .

5. Find the H. C. F. of:

(i) $15a^2 - 60b^2$, $3a^2 - 3ab - 18b^2$, $6a^2 + 2ab - 20b^2$.

(ii) $16a - 15(1 + a)(1 - a)$, $(3a + 1)^2 - 4(a - 2)^2$, $5a^3 - 3a^2$,
 $-10a + 6$

6. Simplify $\frac{x^2 - 13x + 40}{x^2 - 5x + 4} \times \frac{x^2 - 4x}{x^2 - 2x - 48} \div \frac{x^2 - 5x}{x^2 + 5x - 6}$.

PAPER 7

1. State in words $(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$; apply it in expanding $(3x - 2y + 4z)^2$, and simplify $(2p - q + r)^2 - (p + 2q - r)^2 - (2r + q - p)^2$.

2. Find the value of $a^2 + b^2 + c^2$ when $a + b + c = 12$ and $ab + ac + bc = 47$.

3. (i) Find the number which is half-way between $a - b$ and $b - a$.

(ii) Given any three consecutive numbers; prove that the difference of the squares of the greatest and the least is equal to four times the other number.

4. Factorise (i) $81a^4 + 64b^4$, (ii) $35(x-y)^2 - 41(x-y) + 12$.
 (iii) $4x^2 - 9y^2 - 6x - 9y$,
 (iv) $(2x+1)(2x+3)(2x+5)(2x+7)+16$.
5. Find the L. C. M. of $12a^2 + 8a - 15$, $6a^2 + 7a - 3$ and $18a^2 - 21a + 5$.
6. Find the value of x which will make the expression $x^5 - 8x^3 + 11x^2 + 7x - 1789$ exactly divisible by $x^2 + 7x - 1$.

PAPER 8

1. State in words $(a+b+c)(a^2+b^2+c^2-ab-ac-bc) = a^3+b^3+c^3-3abc$; apply it in finding the product of $(2m-n-3)(4m^2+n^2+2mn+6m-3n+9)$.
2. Use the identity $a^2-b^2=(a+b)(a-b)$ to find the difference between the squares of 587047 and 312953.
3. Resolve into factors :
 (i) $512(x-\frac{1}{8})^3 - 8(x-\frac{1}{2})^3$,
 (ii) $1+2x+x(x+2)+(x+1)(x+2)$,
 (iii) $4a^2+12ab+9b^2-8a-12b$,
 (iv) $(a^2-1)(a+2)+(a^2+2a)(a+1)$.
4. Shew by means of a formula that $(ax+by+cz)^3+(cx-by+az)^3$ is divisible by $(a+c)(x+z)$.
5. Simplify $\frac{a^4+a^2b^2+b^4}{a^2-4ab-21b^2} \times \frac{a^2+2ab-3b^2}{a^3-b^3} \div \frac{1}{a-7b}$.
6. Find the value of $27 \times 51 + 27 \times 49 + 73 \times 51 + 73 \times 49$.

PAPER 9

1. (i) Simplify by factors $\frac{.254 \times .254 - .16 \times .16}{.094}$.
 (ii) Write by inspection the continued product of $(x-a)$, (x^2+a^2) , (x^4+a^4) .
2. If $2s=a+b+c$, prove that

$$(s-a)^3+(s-b)^3+(s-c)^3-3(s-a)(s-b)(s-c) = \frac{1}{2}(a^3+b^3+c^3-3abc).$$

3. Divide the product of $2m^2 + 11m - 21$ and $3m^2 - 20m - 7$ by $m^2 - 49$.

4. Find the H. C. F. of $x^2 - 7x + 10$ and $4x^3 - 25x^2 + 20x + 25$.

5. Factorise (i) $a^4 + 4$. (ii) $a^3 + a^2 - a - 1$.
(iii) $a^8 + a^4 + 1$.

6. Simplify $\frac{6x^2 + x - 1}{2x^2 - 5x - 12} \times \frac{6x^2 + 11x + 3}{9x^2 - 1} \div \frac{2x^2 + 9x + 4}{x^2 - 16}$.

PAPER 10

1. Simplify $\frac{\cdot 71 \times \cdot 71 \times \cdot 71 - \cdot 29 \times \cdot 29 \times \cdot 29}{\cdot 71 \times \cdot 71 + \cdot 71 \times \cdot 29 + \cdot 29 \times \cdot 29}$.

2. Factorise (i) $(a^4 - b^4) - (a + b)^2(a - b)^2 + 2b(a^3 + b^3)$,
(ii) $x^3 - 10x^2 + 31x - 30$.

3. Find the H. C. F. of $x^2 - (a - c)x - ac$ and $x^2 - (a + c)x + ac$.

4. Simplify, $\frac{x^4 + x^2 + 1}{x^2 - 1} + \frac{(x - 1)^3}{x^3 - 1} \div \frac{x^3 + 8x^2 - 9x}{x + 1}$.

5. Write down the continued product of $(x - y)$, $(x + y)$, $(x^2 - xy + y^2)$, $(x^2 + xy + y^2)$, and $(x^6 + y^6)$.

6. If $x = a^2 - bc$, $y = b^2 - ca$, $z = c^2 - ab$, prove that $x^3 + y^3 + z^3 - 3xyz = (a^3 + b^3 + c^3 - 3abc)^2$.

CHAPTER VII

H. C. F. BY DIVISION

The H. C. F. of $x^3 - y^3$ and $x^2 - y^2$ is obviously $x - y$. If we multiply the first expression by a^3 or $(a^3 - b^3)$ and the second by a^2 or $a^2 - b^2$, the H. C. F. of the new expressions is $a^2(x - y)$ or $(a - b)(x - y)$ and is thus altered.

This alteration is due to the fact that the multipliers contain a common factor.

If we multiply the first expression by a^3 or $(a^3 - b^3)$ and the second by m^2 or $(m^2 - n^2)$, the H. C. F. of the new expressions is the same as that of the original.

If the original expressions be $(a^3 - b^3)(x^3 - y^3)$ and $x^2 - y^2$, the H. C. F. is $x - y$, and if the second expression be multiplied by $(a - b)$ or $a^2 - b^2$, the H. C. F. is altered. This alteration is due to the fact that the multiplier is either contained in the first expression as a factor or contains a factor which is also a factor of the first expression.

This can be briefly expressed as follows :

The H. C. F. of two expressions A and B is the same as the H. C. F. of mA and nB provided that m does not contain a factor which is contained in B or n , and n does not contain a factor which is contained in A or m .

Again, if the original expressions be $p^3(x^3 - y^3)$ and $q^2(x^2 - y^2)$ and if the first expression be divided by p^3 and the second by q^2 , the H. C. F. remains unaltered, because p^3 and q^2 do not contain a common factor. If the original expressions be $p^3(x^3 - y^3)$ and $p^2(x^2 - y^2)$ and if the first be divided by p^3 and the second by p^2 , the H. C. F. is altered, for p^3 and p^2 do contain p^2 as the common factor.

If A and B stand for two expressions whose factors are $a.H$ and $b.H$ respectively, (where a and b are prime to one another), then obviously $A \pm B = H(a \pm b)$, or the H.C.F. of two expressions is contained in their *sum* as well as *difference*.

Again, since $mA \pm nB = H(ma \pm nb)$, the H. C. F. of two expressions is contained in the sum and difference of their multiples, provided that the multiplying factors (m and n) satisfy the conditions laid down above.

Example 1. Find the H.C.F. of $a^3 + a^2 - 2$ and $a^3 + 2a^2 - 3$.

The process adopted in finding out H. C. F. is similar to one used in finding out the G. C. M. of large numbers in arithmetic.

a	$a^3 + a^2 - 2$ $a^3 - a$	$a^3 + 2a^2 - 3$ $a^3 + a^2 - 2$	1
1	$a^2 + a - 2$ $a^2 - 1$	$a^2 - 1$ $a^2 - a$	a
	$a - 1$	$a - 1$ $a - 1$	1

\therefore the H. C. F. $= a - 1$.

Steps in the process. Divide $a^3 + 2a^2 - 3$ by $a^3 + a^2 - 2$, the remainder $= a^2 - 1$. This remainder contains the required H. C. F.

Divide $a^3 + a^2 - 2$ by $a^2 - 1$, the remainder in this case is $a - 1$. This remainder contains the required H. C. F.

Divide $a^2 - 1$ by $a - 1$. Since in this case there is no remainder, $a - 1$ is the H. C. F.

Example 2. Find the H. C. F. of $a^5 - a^3 - 4a^2 - 3a - 2$ and $2a^5 - a^4 - a^3 - 3a^2$.

The second expression $= a^2(2a^3 - a^2 - a - 3)$.

Since a^2 is a factor of the second expression but not of the first, therefore it may be neglected. The second expression thus left $= 2a^3 - a^2 - a - 3$.

Now the first expression, as it is, cannot be divided by $2a^3 - a^2 - a - 3$, unless we multiply it by 2, which is not a factor of the second expression.

Thus we have

		a^5	$- a^3 - 4a^2 - 3a - 2$	
		2		
	$2a^3 - a^2 - a - 3$	$2a^5$	$- 2a^3 - 8a^2 - 6a - 4$	
-2	$2a^3 + 18a^2 + 18a + 16$	$2a^5 - a^4 - a^3 - 3a^2$		a^2
-19	$-19a^2 - 19a - 19$	$a^4 - a^3 - 5a^2 - 6a - 4$		
	$a^2 + a + 1$	2		
		$2a^4 - 2a^3 - 10a^2 - 12a - 8$		
		$2a^4 - a^3 - a^2 - 3a$		a
		$- a^3 - 9a^2 - 9a - 8$		
		$- a^3 - a^2 - a$		$-a$
		$- 8a^2 - 8a - 8$		
		$- 8a^2 - 8a - 8$		-8

\therefore the H. C. F. $= a^2 + a + 1$.

NOTE. The method illustrated above can be used with special advantage, where the expressions cannot be easily factorised. In this method it is *necessary* to arrange the terms in the expressions according to ascending or descending powers of a certain letter.

EXERCISE 46.

Find by the method of division the H. C. F. of :

- | | |
|------------------------------------|--------------------------------------|
| 1. $a^3 + 6a^2 + 13a + 12,$ | $a^3 + 7a^2 + 16a + 16.$ |
| 2. $a^3 + 7a^2 + 17a + 15,$ | $a^3 + 8a^2 + 19a + 12.$ |
| 3. $a^3 - 10a^2 + 26a - 8,$ | $a^3 - 9a^2 + 23a - 12.$ |
| 4. $6a^3 - 5a^2 + 4a - 1,$ | $3a^3 + 2a^2 + 5a - 2.$ |
| 5. $2a^3 - 7a^2 - 8a - 35,$ | $2a^3 + 9a^2 + 16a + 21.$ |
| 6. $3a^3 + 4a^2 - 6a - 8,$ | $36a^3 + 27a^2 - 16a + 16.$ |
| 7. $24a^4 - 72a^3 + 54a^2,$ | $16a^5 - 48a^4 + 36a^3.$ |
| 8. $2a^3 + 5a^2 + a - 2,$ | $2a^3 - 5a^2 - 4a + 3.$ |
| 9. $a^3 + 4a^2b - 8ab^2 + 24b^3,$ | $a^5 - a^4b + 8a^2b^3 - 8ab^4.$ |
| 10. $a^3 - 4a^2 + 2a + 3,$ | $2a^4 - 9a^3 + 12a^2 - 7.$ |
| 11. $a^4 + a^2 - 6,$ | $a^4 - 3a^2 + 2.$ |
| 12. $a^4 - 9a^2 - 30a - 25,$ | $a^5 + a^4 - 7a^2 + 5a.$ |
| 13. $2a^4 - 6a^3 + 3a^2 - 3a + 1,$ | $a^7 - 3a^6 + a^5 - 4a^2 + 12a - 4.$ |

14. $2a^4 + 9a^3 + 14a + 3$, $3a^4 + 15a^3 + 5a^2 + 10a + 2$.
 15. $12a^4 - 30a^3 + 156a - 210$, $15a^4 - 25a^3 + 145a - 75$.

To find the H. C. F. of more than two expressions by this method, first we find the H. C. F. of two of the expressions, and then find the H. C. F. of this H. C. F. and the third expression; and so on. The last H. C. F. is the one required.

Find by the method of division the H. C. F. of:

16. $x^3 - 4x^2 + 4x$, $x^3 + x^2 - 7x + 2$, $2x^3 - x^2 - 7x + 2$.
 17. $x^3 - x^2 - 7x + 3$, $x^3 - 5x^2 + 9x - 9$, $x^3 - 4x^2 + 4x - 3$.
 18. $2x^3 - 7x^2 + 7x - 2$, $4x^3 - 13x^2 + 11x - 2$, $x^4 - 3x^3 + 6x - 4$.
 19. $x^3 + 6x^2y + 9xy^2 + 4y^3$, $x^4 + 9x^3y + 28x^2y^2 + 36xy^3 + 16y^4$
 and $x^4 + 8x^3y + 21x^2y^2 + 22xy^3 + 8y^4$.
 20. $3x^3 + 28x^2y + 52xy^2 - 48y^3$, $3x^3 + 4x^2y - 28xy^2 + 16y^3$
 and $3x^3 + 10x^2y - 44xy^2 + 24y^3$.

In certain cases the questions on H. C. F. can be done more easily by the *alternate destruction of the highest and the lowest terms*, as illustrated in the next example:

Example 3. Find the H. C. F. of $x^5 - x^3 + 8$ and $x^5 - x^2 + 4$

Let $A = x^5 - x^3 + 8$

and $B = x^5 - x^2 + 4$.

Then $A - B = -x^3 + x^2 + 4$ [Divide by -1 and call the result C .]

or $C = x^3 - x^2 - 4$.

Again, $2B - A = x^5 + x^3 - 2x^2$, [Divide by x^2 and call the result D .]

or $D = x^3 + x - 2$.

Now the H. C. F. of A and B is the same as that of C and D .

$D - C = x^2 + x + 2$. [Call it E .]

$C - 2D = -x^3 - x^2 - 2x$, [Divide by $-x$ and call the result F .]

or $F = x^2 + x + 2$.

Now the H. C. F. of E and F is the same as that of C and D or that of A and B , but E and F are identical, therefore $x^2 + x + 2$ is the required H. C. F.

NOTE. This method is of universal application and *especially useful* when the co-efficients of the given expressions are cross-wise equal or related as multiples and sub-multiples.

Find by the method of alternate destruction of the highest and the lowest terms, the H. C. F. of :

21. $2x^4 + x^3 - 6x^2 - 2x + 3$ and $2x^4 - 3x^3 + 2x - 3$.
22. $x^4 - 3x + 20$ and $5x^4 - 3x^3 + 64$.
23. $x^4 + x^3 - 5x^2 - 3x + 2$ and $x^4 - 3x^3 + x^2 + 3x - 2$.
24. $2x^5 - 11x^2 - 9$ and $4x^5 + 11x^4 + 81$.
25. $2x^5 - 5x^2 + 3$ and $3x^5 - 5x^3 + 2$.
26. $x^5 + 11x^3 - 54$ and $x^5 + 11x + 12$.
27. $6x^3 - 11x^2 + 5x - 3$ and $9x^3 - 9x^2 + 5x - 2$.

Example 4. Reduce to the lowest terms $\frac{x^3 + x^2 + x - 3}{x^3 + 3x^2 + 5x - 3}$.

The H. C. F. of the numerator and the denominator when found by the *division method* or by the *method of alternate destruction of the highest and the lowest terms* is equal to $x^2 + 2x + 3$.

$$\begin{aligned} \therefore \text{the expression} &= \frac{(x^3 + x^2 + x - 3) \div (x^2 + 2x + 3)}{(x^3 + 3x^2 + 5x - 3) \div (x^2 + 2x + 3)} \\ &= \frac{x - 1}{x + 1}. \end{aligned}$$

Reduce to the lowest terms :

28. $\frac{2x^2 - 13x + 18}{2x^3 - 11x^2 + 3x + 27}$
29. $\frac{x^3 - 3x + 2}{x^3 + 3x^2 - 4}$
30. $\frac{x^3 - 13x + 12}{x^3 + x - 2}$
31. $\frac{4x^3 - 8x^2 + 5x - 3}{12x^3 + 4x^2 + x + 5}$
32. $\frac{x^4 - 4x^3 + 4x^2 - 4x + 3}{x^4 - 2x^3 - 2x^2 - 2x - 3}$
33. $\frac{x^4 - 13x^2 + 36}{x^4 - x^3 - 7x^2 + x + 6}$
34. $\frac{3x^4 - 14x^3 - 9x + 2}{2x^4 - 9x^3 - 14x + 3}$
35. $\frac{x^4 - 2x^3 - 25x^2 + 26x + 120}{x^4 - 4x^3 - 19x^2 + 46x + 120}$

CHAPTER VIII

L. C. M. by H. C. F.

Let A and B stand for two expressions whose H. C. F. is H . Divide A and B by H and let a and b be their respective quotients.

$$\text{Then } \left. \begin{array}{l} A = a \cdot H \\ B = b \cdot H \end{array} \right\}$$

Since a and b have no common factor, the L. C. M. of A and B is obviously

$$= a \cdot H \cdot b.$$

Let L stand for the L. C. M. of A and B ,

$$\text{then } L = a \cdot H \cdot b, \quad (i)$$

$$\text{or } L \cdot H = (a \cdot H) \cdot (b \cdot H) \\ = A \cdot B. \quad (ii)$$

$$\therefore \left. \begin{array}{l} L = \frac{A}{H} \cdot B \\ \text{or } L = A \cdot \frac{B}{H} \end{array} \right\} \quad (iii)$$

From (i), (ii) and (iii) we establish three important rules :

(i) *To find the L. C. M. of two expressions, we have to divide each by their H. C. F. and multiply the product of the quotients thus got by the H. C. F.*

(ii) *The product of the L. C. M. and the H. C. F. of two expressions is equal to the product of the expressions.*

(iii) *To find the L. C. M. of two expressions we have to divide one of them by their H. C. F. and multiply the quotient by the other.*

Example 1. Find the L. C. M. of $x^3 - 9x^2 + 26x - 24$ and $x^3 - 6x^2 + 11x - 6$.

First we find the H. C. F. of these expressions.

x	$x^3 - 6x^2 + 11x - 6$	$x^3 - 9x^2 + 26x - 24$	1
	$x^3 - 5x^2 + 6x$	$x^3 - 6x^2 + 11x - 6$	
-1	$-x^2 + 5x - 6$	$-3x^2 + 15x - 18$	
	$-x^2 + 5x - 6$	$x^2 - 5x + 6$	-3

\therefore the H.C.F. $= x^2 - 5x + 6$ and when the second expression is divided by the H. C. F., the quotient $= x - 1$.

\therefore the L. C. M. $= (x - 1)(x^3 - 9x^2 + 26x - 24)$.

Example 2. The H. C. F. of two expressions is $x - 7$ and their L. C. M. is $x^3 - 10x^2 + 11x + 70$. One of the expressions is $x^2 - 5x - 14$. Find the other.

$\therefore (x - 7)(x^3 - 10x^2 + 11x + 70) = (x^2 - 5x - 14)(2\text{nd expression})$,

$$\begin{aligned} \therefore \text{the 2nd expression} &= \frac{(x - 7)(x^3 - 10x^2 + 11x + 70)}{x^2 - 5x - 14} \\ &= \frac{(x - 7)(x^3 - 10x^2 + 11x + 70)}{(x - 7)(x + 2)} \\ &= \frac{x^3 - 10x^2 + 11x + 70}{x + 2} \\ &= x^2 - 12x + 35. \end{aligned}$$

EXERCISE 47.

Find the H.C.F. and L.C.M. of the following expressions :

1. $x^3 - 5x^2 + 9x - 9$, $x^3 + x^2 - 3x + 9$.
2. $x^3 - 2x^2 - 13x - 10$, $x^3 - x^2 - 10x - 8$.
3. $4x^3 - 10x^2 - 18x + 45$, $6x^3 + 8x^2 - 27x - 36$.
4. $2x^4 + x^3 - 9x^2 + 8x - 2$, $2x^4 - 7x^3 + 11x^2 - 8x + 2$.
5. $x^4 + x^3 - 9x^2 - 3x + 18$, $x^5 + 6x^2 - 49x + 42$.
6. $8x^4 - 6x^3 - 8x^2 + 9x - 6$, $16x^4 - 12x^3 + 20x^2 - 9x + 6$.
7. $x^4 - 2x^3 + 5x^2 - 4x + 3$, $2x^4 - x^3 + 6x^2 + 2x + 3$.
8. $15x^3 - 31ax^2 + 5a^2x + 2a^3$, $6x^4 - 25ax^3 + 26a^2x^2 - a^4$.

9. The H. C. F. of two expressions is $5x^2 - 2x - 1$, their L. C. M. is $30x^4 + 13x^3 - 11x^2 - 7x - 1$, and one expression is $10x^3 + x^2 - 4x - 1$; find the other.

10. The H. C. F. of two expressions is $x^3 + 1$, their L. C. M. is $(x^3 + 1)(3x - 1)(4x - 1)$, and one expression is $3x^4 - x^3 + 3x - 1$; find the other.

11. The H. C. F. of two expressions is $1 + x^3 - x^4$, their L. C. M. is $(1 + x)(1 - x^3)(1 + x^3 - x^4)$, and one expression is $1 + x + x^3 - x^5$; find the other.

12. The H. C. F. of two expressions is $x^2 - 2x + 4$, their L. C. M. is $x^6 - 2x^5 + 7x^4 - 6x^3 - 6x^2 + 36x - 72$, and one expression is $x^4 - 2x^3 + x^2 + 6x - 12$; find the other.

To find the L. C. M. of more than two expressions, first we find the L. C. M. of two expressions, then the L. C. M. of that L. C. M. and the third expression, and so on. The last L. C. M. is the one required.

Find the L. C. M. of the following :

13. $x^3 + x + 2$, $x^3 + 3x + 4$ and $x^3 + x^2 + 4$.

*14. $x^5 + x^3 + x^2 + 1$, $x^4 - x^3 + x - 1$ and $x^5 + 2x^4 - x - 2$.

*15. $x^5 + x^4 - 4x^3 + 2x^2 + 6x - 9$, $x^4 - x^2 + 6x - 9$ and
 $x^4 + 2x^3 - 5x^2 - 6x + 9$.

CHAPTER IX

FRACTIONS

1. To reduce fractions to equivalent fractions with a common denominator.

Example 1. Reduce $\frac{2x}{a^2(a+x)}$, $\frac{3y}{b^2(a-x)}$, $\frac{4z}{c^2(a^2-x^2)}$ to equivalent fractions with the lowest common denominator.

The L. C. M. of the denominators $= a^2b^2c^2(a+x)(a-x)$ and the quotients obtained on dividing it by the denominators are $b^2c^2(a-x)$, $a^2c^2(a+x)$ and a^2b^2 respectively.

Hence we have

$$\begin{aligned}\frac{2x}{a^2(a+x)} &= \frac{b^2c^2(a-x)2x}{a^2b^2c^2(a+x)(a-x)} = \frac{2b^2c^2x(a-x)}{a^2b^2c^2(a+x)(a-x)}, \\ \frac{3y}{b^2(a-x)} &= \frac{a^2c^2(a+x)3y}{a^2b^2c^2(a+x)(a-x)} = \frac{3a^2c^2y(a+x)}{a^2b^2c^2(a+x)(a-x)}, \\ \text{and } \frac{4z}{c^2(a^2-x^2)} &= \frac{a^2b^2(4z)}{a^2b^2c^2(a+x)(a-x)} = \frac{4a^2b^2z}{a^2b^2c^2(a+x)(a-x)}.\end{aligned}$$

EXERCISE 48.

Reduce the following to the equivalent fractions with the lowest common denominator:

$$1. \quad \frac{1}{x+1}, \quad \frac{3}{4x+4}, \quad \frac{x}{x^2-1} \quad 2. \quad \frac{3}{x^2-2x-3}, \quad \frac{4}{x^2-5x+6}$$

$$3. \quad \frac{a}{a-b}, \quad \frac{b}{a+b}, \quad \frac{ab}{a^2-b^2}, \quad \frac{b^2}{a^2+b^2}.$$

$$4. \quad \frac{1}{x^2-ax+a^2}, \quad \frac{1}{x^2+ax+a^2}, \quad \frac{a^2}{x^4+a^2x^2+a^4}.$$

$$5. \quad \frac{a}{a-b}, \quad \frac{a+b}{a^2+ab+b^2}, \quad \frac{ab}{a^3-b^3}.$$

$$6. \frac{1}{x^2 - (a+b)x + ab}, \frac{1}{x^2 - (a+c)x + ac}, \frac{1}{x^2 - (b+c)x + bc}.$$

$$7. \frac{1}{(a-b)^2 - c^2}, \frac{1}{(b-c)^2 - a^2}, \frac{1}{(c-a)^2 - b^2}.$$

$$8. \frac{a}{b(a-b-c)}, \frac{b}{a(a-b+c)}, \frac{c}{a^2 + b^2 - c^2 - 2ab}.$$

Example 2. Reduce to the lowest common denominator

$$\frac{a}{x-a}, \frac{x}{a-x}, \frac{a^2}{x^2-a^2}, \frac{ax}{a^2-x^2}.$$

Here we have to arrange the denominators in the same order.

$$\text{Thus } \frac{x}{a-x} = \frac{-1 \times x}{-(a-x)} = \frac{-x}{x-a},$$

$$\text{and } \frac{ax}{a^2-x^2} = \frac{-1 \times ax}{-(a^2-x^2)} = \frac{-ax}{x^2-a^2}.$$

The L. C. M. of the denominators when arranged as above $= x^2 - a^2$ and the quotients obtained by dividing it by the denominators are $x+a$, $x+a$, 1, and 1 respectively.

Hence we have

$$\frac{a}{x-a} = \frac{a(x+a)}{x^2-a^2},$$

$$\frac{x}{a-x} = \frac{-x}{x-a} = \frac{-x(x+a)}{x^2-a^2},$$

$$\frac{a^2}{x^2-a^2} \text{ stands as it is,}$$

$$\frac{ax}{a^2-x^2} = \frac{-ax}{x^2-a^2}.$$

Reduce to the lowest common denominator :

$$9. \frac{2}{a-b}, \frac{3}{b-a}, \frac{4}{a+b}.$$

$$10. \frac{a+b}{a-b}, \frac{a-b}{a+b}, \frac{2ab}{b^2-a^2}.$$

$$11. \frac{4a-b}{1-4ab}, \frac{4a+b}{1+4ab}, \frac{4b(1-8a^2)}{16a^2b^2-1}.$$

$$12. \frac{3}{x^2-5x+6}, \frac{4}{x^2-4x+3}, \frac{5}{3x-2-x^2}.$$

2. Addition and Subtraction of Fractions.

Example 1. Find the value of $\frac{x}{x-y} + \frac{y}{y-x}$.

$$\text{Since } \frac{y}{y-x} = \frac{y \times (-1)}{(y-x) \times (-1)} = \frac{-y}{x-y},$$

$$\begin{aligned} \text{we have } \frac{x}{x-y} + \frac{y}{y-x} &= \frac{x}{x-y} + \frac{-y}{x-y} \\ &= \frac{x+(-y)}{x-y} = \frac{x-y}{x-y} = 1. \end{aligned}$$

Example 2. Find the value of $\frac{x}{x-y} - \frac{y}{x+y}$.

The L. C. M. of the denominators $= x^2 - y^2$.

$$\begin{aligned} \therefore \frac{x}{x-y} - \frac{y}{x+y} &= \frac{x(x+y)}{x^2-y^2} - \frac{y(x-y)}{x^2-y^2} \\ &= \frac{x(x+y) - y(x-y)}{x^2-y^2} \\ &= \frac{x^2 + y^2}{x^2 - y^2}. \end{aligned}$$

Example 3. Simplify $\frac{1}{x^2-5x+6} - \frac{1}{x^2-6x+8} - \frac{2}{x^2-7x+12}$

Factorising the denominators, we have

$$\text{the expression} = \frac{1}{(x-2)(x-3)} - \frac{1}{(x-2)(x-4)} - \frac{2}{(x-3)(x-4)}$$

The L. C. M. of the denominators is $(x-2)(x-3)(x-4)$

$$\begin{aligned} \therefore \text{The given expression} &= \frac{(x-4) - (x-3) - 2(x-2)}{(x-2)(x-3)(x-4)} \\ &= \frac{x-4-x+3-2x+4}{(x-2)(x-3)(x-4)} \\ &= \frac{-2x+3}{(x-2)(x-3)(x-4)}. \end{aligned}$$

EXERCISE 49.

Find the value of :

1. $\frac{2}{x-2} + \frac{3}{x+3}$

2. $\frac{4}{x-3} + \frac{5}{x+4}$

3. $\frac{x+a}{x-a} - \frac{x-a}{x+a}$

4. $1 + \frac{4xy}{(x-y)^2}$

$$5. \frac{x-1}{x+1} - \frac{x-2}{x+2}$$

$$6. (1+x) + \frac{x^2}{1-x}$$

$$7. \frac{1}{a^2+ab} + \frac{1}{b^2+ab}$$

$$8. \frac{x}{x^2-y^2} - \frac{y}{y^2-x^2}$$

$$9. \frac{1}{2x+3y} - \frac{(2x-3y)^2}{8x^3+27y^3}$$

$$10. \frac{1}{(a-b)(a-c)} + \frac{1}{(a-c)(b-c)}$$

$$11. \frac{a+b}{a-b} - \frac{a-b}{a+b} + \frac{2ab}{b^2-a^2}$$

$$12. \frac{1}{a^2-b^2} - \frac{2}{(a+b)^2} + \frac{3}{(a-b)^2}$$

$$13. \frac{1}{x^2-5x+6} + \frac{1}{x^2-9x+20}$$

$$14. \frac{1}{x^2+3x-10} - \frac{1}{2x^2-x-6}$$

$$15. \frac{1}{x^2-3x+2} + \frac{1}{x^2-5x+6} + \frac{2}{x^2-8x+15}$$

$$16. \frac{1}{x^2+7x+12} + \frac{1}{x^2+9x+20} - \frac{2}{x^2+8x+15}$$

$$17. \frac{1}{x^2+3x+2} + \frac{1}{x^2+4x+3} - \frac{2}{x^2+5x+6}$$

$$18. \frac{2x-3}{x^2+2x-3} - \frac{3x-4}{x^2+x-2} + \frac{x+5}{x^2+5x+6}$$

$$19. \frac{1}{x^2+5ax+4a^2} + \frac{1}{x^2+11ax+28a^2} + \frac{2}{x^2+20ax+91a^2}$$

Example 4: Simplify $\frac{2a^2+a-1}{a+1} + \frac{3a^2+5a+2}{3a+2} + \frac{4-a^2}{a+2}$.

Factorising the numerators of these fractions, we have the expression

$$= \frac{(2a-1)(a+1)}{a+1} + \frac{(3a+2)(a+1)}{3a+2} + \frac{(2-a)(2+a)}{a+2}$$

$$= (2a-1) + (a+1) + (2-a)$$

$$= 2a+2 = 2(a+1).$$

Simplify:

$$20. \frac{x^2-1}{x-1} + \frac{x^2-7x+12}{x-4} + \frac{x^2+5x+6}{x+3}$$

$$21. \frac{x^2-2x-3}{x^2-4x+3} + \frac{5x^2+5x-30}{x^2+2x-3} + \frac{x^2+x-2}{x^2-2x+1}$$

$$22. \frac{a^2 - (2b - 3c)^2}{(3c + a)^2 - 4b^2} + \frac{4b^2 - (3c - a)^2}{(a + 2b)^2 - 9c^2} + \frac{9c^2 - (a - 2b)^2}{(2b + 3c)^2 - a^2}.$$

$$23. \frac{a^4 + a^2b^2 + b^4}{a^3 + b^3} - \frac{a^2 + ab}{a - b} + \frac{b^3}{a^2 - b^2}.$$

$$24. \frac{a^2 - 1}{(a + 1)^2} + \frac{a^2 - 5a + 6}{a^2 - 4a + 4} + \frac{1}{a + 1}.$$

Example 5. Find the value of $\frac{x^3 + 2x + 3}{x + 1} - \frac{x^3 - 2x + 1}{x - 1}$

Since it is not so easy to factorise the numerators, we actually divide each numerator by the corresponding denominator,

$$\begin{aligned} \text{The expression} &= (x^2 - x + 3) - (x^2 + x - 1) \\ &= x^2 - x + 3 - x^2 - x + 1 \\ &= -2x + 4 = 2(2 - x). \end{aligned}$$

$$25. \text{ Find the value of } \frac{x^3 + 3x^2 - 4}{x + 2} + \frac{x^3 + 1}{x + 1} + \frac{2x^3 + 6x^2 + x + 3}{x + 3}.$$

$$26. \text{ Find the value of } \frac{3x^3 - 4x^2 - 2x + 6}{x^2 - 1} - \frac{3x^3 - 4x^2 + 4x - 3}{x^2 + 1}.$$

Example 6. Simplify $\frac{1}{a^2 - 1} + \frac{1}{a^2 - 2} - \frac{1}{a^2 + 1} - \frac{1}{a^2 + 2}.$

Combining fractions having almost similar denominators, we have the expression

$$\begin{aligned} &= \left(\frac{1}{a^2 - 1} - \frac{1}{a^2 + 1} \right) + \left(\frac{1}{a^2 - 2} - \frac{1}{a^2 + 2} \right) \\ &= \frac{(a^2 + 1) - (a^2 - 1)}{a^4 - 1} + \frac{(a^2 + 2) - (a^2 - 2)}{a^4 - 4} \\ &= \frac{2}{a^4 - 1} + \frac{4}{a^4 - 4} \\ &= \frac{2(a^4 - 4) + 4(a^4 - 1)}{(a^4 - 1)(a^4 - 4)} \\ &= \frac{6(a^4 - 2)}{(a^4 - 1)(a^4 - 4)}. \end{aligned}$$

Example 7. Simplify $\frac{1}{a+3} + \frac{1}{a-3} + \frac{6}{9-a^2}$.

Since the first two fractions have almost similar denominators, we combine them together and change the signs of the third.

$$\begin{aligned} \therefore \text{the expression} &= \left(\frac{1}{a+3} + \frac{1}{a-3} \right) - \frac{6}{a^2-9} \\ &= \frac{a-3+a+3}{a^2-9} - \frac{6}{a^2-9} \\ &= \frac{2a}{a^2-9} - \frac{6}{a^2-9} \\ &= \frac{2(a-3)}{a^2-9} = \frac{2}{a+3} \end{aligned}$$

Example 8. Simplify $\frac{1}{a-1} - \frac{a}{a^2-1} - \frac{a^2}{a^4-1} - \frac{a^4}{a^8-1}$.

In exercises of this type, it is useful to combine the first fraction with the second, the result so obtained with the third, and so on.

$$\begin{aligned} \text{The expression} &= \left(\frac{1}{a-1} - \frac{a}{a^2-1} \right) - \frac{a^2}{a^4-1} - \frac{a^4}{a^8-1} \\ &= \frac{(a+1)-a}{a^2-1} - \frac{a^2}{a^4-1} - \frac{a^4}{a^8-1} \\ &= \frac{1}{a^2-1} - \frac{a^2}{a^4-1} - \frac{a^4}{a^8-1} \\ &= \left(\frac{1}{a^2-1} - \frac{a^2}{a^4-1} \right) - \frac{a^4}{a^8-1} \\ &= \frac{(a^2+1)-a^2}{a^4-1} - \frac{a^4}{a^8-1} \\ &= \frac{1}{a^4-1} - \frac{a^4}{a^8-1} \\ &= \frac{a^4+1-a^4}{a^8-1} = \frac{1}{a^8-1} \end{aligned}$$

Simplify:

27. $\frac{1}{a-1} - \frac{1}{a+1} + \frac{1}{a-2} - \frac{1}{a+2}$.

$$28. \frac{1}{a-b} - \frac{2}{2a+b} + \frac{1}{a+b} - \frac{2}{2a-b}$$

$$29. \frac{1}{a-3b} + \frac{3}{a+b} + \frac{1}{a+3b} + \frac{3}{a-b}$$

$$30. \frac{1}{a^2-1} + \frac{1}{a^2-3} - \frac{1}{a^2+1} - \frac{1}{a^2+3}$$

$$31. \frac{3}{1+a} - \frac{2}{1-a} - \frac{5a}{a^2-1} \quad 32. \frac{3-x}{1-3x} - \frac{3+x}{1+3x} - \frac{1-16x}{9x^2-1}$$

$$33. \frac{a+b}{a-b} - \frac{2ab}{a^2-b^2} - \frac{2a^2b^2}{a^4-b^4}$$

$$34. \frac{1}{1-a} + \frac{1}{1+a} + \frac{2}{1+a^2} + \frac{4}{1+a^4}$$

$$35. \frac{1}{a-1} - \frac{1}{a+1} + \frac{a-2}{a^2-a+1} - \frac{a+2}{a^2+a+1}$$

$$36. \frac{1}{a+b} + \frac{3b}{ab-a^2} + \frac{4b-a}{a^2-b^2}$$

$$37. \frac{1}{x+a} + \frac{a}{x^2-a^2} + \frac{x}{x^2+a^2} + \frac{2x^3}{x^4+a^4}$$

$$38. \frac{1}{a-1} - \frac{a}{a^2-1} - \frac{a^2}{a^4-1} - \frac{a^4}{a^8-1}$$

$$39. \frac{1}{x-a} + \frac{2x}{x^2+a^2} + \frac{4x^3}{x^4+a^4} - \frac{8x^7}{x^8-a^8}$$

[Hint. Add and subtract $\frac{1}{x+a}$.]

$$40. \frac{a-b}{a^2-ab+b^2} + \frac{a+b}{a^2+ab+b^2} + \frac{2b^3}{a^4+a^2b^2+b^4}$$

$$41. \frac{1}{a^2-ab+b^2} - \frac{2ab}{a^4-a^2b^2+b^4} + \frac{4a^3b^3}{a^8-a^4b^4+b^8} - \frac{8a^7b^7}{a^{16}+a^8b^8+b^{16}}$$

$$42. \frac{1}{x+3} + \frac{x+1}{x^2-3x+9} - \frac{2x^2+x+12}{x^3+27}$$

$$43. \frac{1}{(n-1)n} + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)}$$

Example 9. Simplify $\frac{1}{x+1} - \frac{3}{x+2} + \frac{3}{x+3} - \frac{1}{x+4}$.

Combining the first and the fourth terms and the second and the third, we have the expression

$$\begin{aligned}
 &= \left(\frac{1}{x+1} - \frac{1}{x+4} \right) - 3 \left(\frac{1}{x+2} - \frac{1}{x+3} \right) \\
 &= \frac{x+4-x-1}{(x+1)(x+4)} - 3 \cdot \frac{x+3-x-2}{(x+2)(x+3)} \\
 &= \frac{3}{(x+1)(x+4)} - \frac{3}{(x+2)(x+3)} \\
 &= 3 \left\{ \frac{1}{(x+1)(x+4)} - \frac{1}{(x+2)(x+3)} \right\} \\
 &= 3 \left\{ \frac{(x+2)(x+3) - (x+1)(x+4)}{(x+1)(x+2)(x+3)(x+4)} \right\} \\
 &= \frac{3(x^2 + 5x + 6 - x^2 - 5x - 4)}{(x+1)(x+2)(x+3)(x+4)} \\
 &= \frac{6}{(x+1)(x+2)(x+3)(x+4)}.
 \end{aligned}$$

Simplify :

$$44. \quad \frac{1}{x+2} - \frac{3}{x+3} + \frac{3}{x+4} - \frac{1}{x+5}.$$

$$45. \quad \frac{1}{x+4} - \frac{3}{x+2} + \frac{3}{x} - \frac{1}{x-2}.$$

$$46. \quad \frac{1}{x+1} - \frac{1}{x+3} - \frac{1}{x+5} + \frac{1}{x+7}.$$

$$47. \quad \frac{1}{x+a} - \frac{1}{x+b} + \frac{1}{x-a} - \frac{1}{x-b}.$$

$$48. \quad \frac{1}{a^2-2b^2} - \frac{1}{a^2+2b^2} + \frac{1}{a^2-3b^2} - \frac{1}{a^2+3b^2}.$$

3. Complex and Continued Fractions. The process for the simplification of **complex fractions** in Algebra is similar to the one in Arithmetic. In Algebra, however, the process is generally shortened by the application of factors. A few examples given further illustrate the method of simplification.

Example 1. Simplify $\frac{\frac{a}{b} + \frac{b}{a} - 2}{\frac{1}{b^2} - \frac{1}{a^2}}$.

$$\begin{aligned} \text{The fraction} &= \frac{\frac{a^2 + b^2 - 2ab}{ab}}{\frac{a^2 - b^2}{a^2 b^2}} = \frac{(a-b)^2}{ab} \times \frac{a^2 b^2}{a^2 - b^2} \\ &= \frac{ab(a-b)}{a+b}. \end{aligned}$$

Example 2. Simplify $\frac{\frac{a+b}{a-b} - \frac{a-b}{a+b}}{\frac{a+b}{a-b} + \frac{a-b}{a+b}}$.

$$\frac{a+b}{a-b} - \frac{a-b}{a+b} = \frac{(a+b)^2 - (a-b)^2}{a^2 - b^2} = \frac{4ab}{a^2 - b^2}$$

$$\frac{a+b}{a-b} + \frac{a-b}{a+b} = \frac{(a+b)^2 + (a-b)^2}{a^2 - b^2} = \frac{2(a^2 + b^2)}{a^2 - b^2}.$$

$$\begin{aligned} \text{Hence the given fraction} &= \frac{4ab}{a^2 - b^2} \div \frac{2(a^2 + b^2)}{a^2 - b^2} \\ &= \frac{4ab}{a^2 - b^2} \times \frac{a^2 - b^2}{2(a^2 + b^2)} \\ &= \frac{2ab}{a^2 + b^2}. \end{aligned}$$

EXERCISE 50.

Simplify :

1. $\frac{1 - \frac{1}{1+x}}{\frac{1}{1+x}}$.

3. $\frac{\frac{1+x}{1-x} - \frac{2x}{1+x}}{\frac{1-x}{1+x} + \frac{2x}{1-x}}$.

2. $\frac{\frac{1}{x-1} + \frac{x}{x+1}}{\frac{x}{x-1} - \frac{1}{x+1}}$.

4. $\frac{\frac{x}{x-1} - \frac{x+1}{x}}{\frac{x}{x+1} - \frac{x-1}{x}}$.

$$5. \frac{\frac{x+y}{x-y} - \frac{x^2+y^2}{x^2-y^2}}{\frac{x}{x+y} + \frac{y}{x-y}}$$

$$6. \frac{\frac{x+2}{x+1} - \frac{1}{x+3}}{\frac{1}{x+1} + \frac{x+2}{x+3}}$$

$$7. \frac{\frac{1}{x^2-xy+y^2} + \frac{1}{x^2+xy+y^2}}{\frac{x+y}{x-y} + \frac{x-y}{x+y}}$$

$$8. \frac{x - \frac{2}{x} - 1}{x - \frac{6}{x} + 1}$$

$$9. \frac{\left(\frac{x+a}{x-a}\right)^2 - x^2}{\frac{x+a}{x-a} - x}$$

$$10. \frac{\frac{x^3-y^3}{x-y} + \frac{x^3+y^3}{x+y}}{\frac{x+y}{x-y} - \frac{x-y}{x+y}}$$

$$11. \frac{1 + \frac{x-y}{x+y}}{1 - \frac{x-y}{x+y}} \times \frac{1 + \frac{y-z}{y+z}}{1 - \frac{y-z}{y+z}} \times \frac{1 + \frac{z-x}{z+x}}{1 - \frac{z-x}{z+x}}$$

$$12. \frac{\frac{x-y}{1+xy} + \frac{y-z}{1+yz}}{1 - \frac{(x-y)(y-z)}{(1+xy)(1+yz)}}$$

$$13. \frac{\frac{x+y}{1-xy} - \frac{x+z}{1-xz}}{1 + \frac{(x+y)(x+z)}{(1-xy)(1-xz)}}$$

$$14. \frac{1 + \frac{1}{a} - a + \frac{1}{a} - \frac{2}{a}}{1 - \frac{1}{a} - a - \frac{1}{a} - 1 + \frac{1}{a^2}}$$

$$15. \frac{\frac{a}{a+1} + \frac{a}{a-1} - \frac{4a-1}{a}}{\frac{2}{a^2-1} - \frac{1}{2+\frac{1}{a}}}$$

$$16. \frac{1}{a - \frac{a}{a+1}} + \frac{1}{a + \frac{a}{a-1}} + \frac{1 - \frac{3}{a}}{1 - \frac{a}{3}}$$

$$17. \left(\frac{1}{\frac{1}{a} + \frac{1}{ab}} + \frac{1}{\frac{1}{a} + \frac{1}{b}} \right) \left(\frac{1}{\frac{1}{a} + \frac{1}{ab}} + \frac{1}{\frac{1}{a} - \frac{1}{b}} \right) \div \frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{a^2} - \frac{1}{b^2}}$$

$$18. \left\{ \frac{y + \frac{x-y}{1+xy}}{1 - \frac{(x-y)y}{1+xy}} - \frac{x - \frac{x-y}{1-xy}}{1 - \frac{x(x-y)}{1-xy}} \right\} \div \left(\frac{x}{y} - \frac{y}{x} \right)^2$$

In Algebra, for the simplification of a **continued fraction**, we begin from the bottom and go upward, step by step, as in Arithmetic.

Example 3. Simplify
$$\frac{a}{a - \frac{1}{a - \frac{a}{1+a}}}$$

$$\begin{aligned} \text{The expression} &= \frac{a}{a - \frac{1}{a - \frac{a}{1+a}}} = \frac{a}{a - \frac{1}{\frac{a^2 - a - a}{1+a}}} \\ &= \frac{a}{a - 1 \times \frac{1+a}{a^2}} = \frac{a}{a - \frac{1+a}{a^2}} \\ &= \frac{a}{\frac{a^3 - 1 - a}{a^2}} = \frac{a \times a^2}{a^3 - a - 1} \\ &= \frac{a^3}{a^3 - a - 1} \end{aligned}$$

Simplify :

19.
$$\frac{1}{x - \frac{1}{x + \frac{1}{x - \frac{1}{x}}}}$$

20.
$$\frac{x}{x - \frac{x-1}{1 - \frac{1}{x+1}}}$$

21.
$$2 - \frac{2}{1 - \frac{3}{2 - \frac{3}{1-x}}}$$

22.
$$\frac{1}{1 - \frac{1}{1 + \frac{1}{1 + \frac{2b-a}{a-b}}}}$$

23.
$$\frac{a}{a^2 - \frac{a^3-1}{a + \frac{1}{a+1}}}$$

24.
$$\frac{1}{1 - \frac{1}{a - \frac{1}{b - \frac{1}{a+b - \frac{a^2}{a-b}}}}}$$

$$25. \frac{1+x^2}{1-x+\frac{x^2}{1+x-\frac{x}{1+x}}}$$

$$26. \frac{x}{1-\frac{x}{1+x+\frac{x}{1-x+x^2}}}$$

$$27. \frac{2x^3}{x^2+y^2-\frac{4xy(x-y)}{(x+y)-\frac{(x-y)^2}{x+y}}}$$

$$28. \frac{x}{1+\frac{x}{2-x+\frac{4x}{3-2x}}}$$

$$29. \frac{3}{x+4-\frac{5(x+3)}{x+4-\frac{x^2+4}{x+1}}}$$

$$30. \frac{\frac{x^2}{y-\frac{x^2}{y+\frac{x^2}{y-\frac{x^2}{y}}}}{\frac{x^2}{y+\frac{x^2}{y-\frac{x^2}{y}}}} - \frac{\frac{x^2}{y+\frac{x^2}{y-\frac{x^2}{y}}}}{\frac{x^2}{y-\frac{x^2}{y+\frac{x^2}{y}}}}$$

It is sometimes useful to decompose a given fraction into two parts, one integral and the other fractional. Such decomposition is illustrated by means of the examples given below:

$$\begin{aligned} \text{Example 4. } \frac{x+3}{x-2} &= \frac{(x-2)+5}{(x-2)} = \frac{x-2}{x-2} + \frac{5}{x-2} \\ &= 1 + \frac{5}{x-2}. \end{aligned}$$

$$\begin{aligned} \text{Example 5. } \frac{2x-3}{x+5} &= \frac{2(x+5)-13}{x+5} \\ &= \frac{2(x+5)}{x+5} - \frac{13}{x+5} \\ &= 2 - \frac{13}{x+5}. \end{aligned}$$

Example 6. Prove that $\frac{2x^2-5x-7}{x-3} \equiv 2x+1 - \frac{4}{x-3}$.

$$x-3 \overline{) 2x^2-5x-7} \quad (2x+1$$

$$\underline{2x^2-6x}$$

$$x-7$$

$$\underline{x-3}$$

$$-4$$

By actual division we obtain $2x+1$ as the quotient and -4 as the remainder.

If the numerator be of a degree lower than that of the denominator, even then the decomposition is possible, as illustrated below :

Example 7. Prove that $\frac{3x}{2x^2+1} \equiv 3x-6x^3+12x^5 - \frac{24x^7}{2x^2+1}$

$$\frac{3x}{2x^2+1} \equiv \frac{3x}{1+2x^2}$$

$$\equiv 3x-6x^3+12x^5 - \frac{24x^7}{1+2x^2}$$

[By actual division.]

$$\equiv 3x-6x^3+12x^5 - \frac{24x^7}{2x^2+1}$$

In such cases, the division can be carried on to any number of terms and the process can be stopped at any step by annexing to the quotient the fraction whose numerator is the last remainder and whose denominator is the divisor.

Thus if we had carried the quotient to four terms, in the above example, we would have got

$$\frac{3x}{2x^2+1} \equiv 3x-6x^3+12x^5-24x^7 + \frac{48x^9}{2x^2+1}$$

Express the following fractions in a form partly integral and partly fractional :

31. $\frac{x+7}{x+3}$

32. $\frac{x+7}{x-3}$

33. $\frac{2x+5}{x-1}$

34. $\frac{3x-4}{x+2}$

35. $\frac{3x-5}{x-3}$

36. $\frac{4x-11}{2x-1}$

37. $\frac{4x-9}{2x+1}$

38. $\frac{6x-1}{3x+2}$

39. $\frac{6x-7}{3x-2}$

40. $\frac{x^4-1}{x+a}$

41. $\frac{4x^3+4x^2+8x+1}{2x^2+2x+3}$

FRACTIONS

Prove that

$$42. \quad \frac{1}{1-x} \equiv 1 + x + x^2 + x^3 + \frac{x^4}{1-x}.$$

$$43. \quad \frac{1}{x-1} \equiv \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^3(x-1)}.$$

$$44. \quad \frac{1}{1+2x} \equiv 1 - 2x + 4x^2 - \frac{8x^3}{1+2x}.$$

$$45. \quad \frac{1}{2x+1} \equiv \frac{1}{2x} - \frac{1}{4x^2} + \frac{1}{8x^3} - \frac{1}{8x^3(2x+1)}.$$

$$46. \quad \frac{1+x^2}{(1-x)^2} \equiv 1 + 2x + 4x^2 + 6x^3 + \frac{8x^4 - 6x^5}{(1-x)^2}.$$

$$47. \quad \frac{x+6}{x+4} + \frac{x+2}{x-4} \equiv 2 + \frac{4x}{x^2-16}.$$

$$48. \quad \frac{4x^2+7}{2x-1} + \frac{6x^2-8x+11}{3x-1} \equiv 4x-1 + \frac{42x-17}{6x^2-5x+1}.$$

$$*49. \quad \frac{3x+6}{(x+1)(x+2)(x+3)} \equiv \frac{1}{(x+1)(x+2)} + \frac{1}{(x+1)(x+3)} + \frac{1}{(x+2)(x+3)}$$

$$*50. \quad \frac{3x^2-14}{(x-1)(x-2)(x-3)} \equiv \frac{x+1}{(x-2)(x-3)} + \frac{x+2}{(x-1)(x-3)} + \frac{x+3}{(x-1)(x-2)}.$$

*51. Apply the identity

$\frac{1}{1-x} \equiv 1 + x + x^2 + \frac{x^3}{1-x}$ in finding the approximate value of $\frac{1}{0.9}$ to 3 decimal places.

CHAPTER X

SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE AND PROBLEMS

1. Let us consider the equation $x + y = 7$.

It is not an identity, for it is not satisfied by *any* values of x and y ; e.g., $x = 8$ and $y = 5$ will not satisfy it.

But if we assign any value to x and the corresponding value to y , as given in the following table,

x	0	1	2	3	4	5	6	7	8	9	10
y	7	6	5	4	3	2	1	0	-1	-2	-3

the equation is satisfied. As the number of such pairs of values is unlimited, such an equation is called an **indeterminate equation**.

* Again, if we take another equation $x - y = 3$, we find that it is not an identity, for it is not satisfied for *any* values of x and y ; e.g., $x = 12$, $y = 7$ will not satisfy it.

But if we assign any value to x and the corresponding value to y , as given in the following table,

x	0	1	2	3	4	5	6	7	8	9	10
y	-3	-2	-1	0	1	2	3	4	5	6	7

the equation is satisfied. As the number of such pairs of values is unlimited, this equation is also *indeterminate*.

* It is important to note that in spite of the fact that both the equations $x + y = 7$ and $x - y = 3$ are indeterminate, *when considered separately*, yet there is *one value* of x , i.e., 5 and *one value* of y , i.e., 2 (see the above tables), for which they are *simultaneously* true.

Such equations for which we have to find a common solution are called **Simultaneous Equations**.

⇒ There are three methods of solving simultaneous equations involving two unknowns, as illustrated below :

Example 1. Solve the equations

$$3x + y = 17 \quad \dots \quad (i)$$

$$8x + 11y = 37 \quad \dots \quad (ii)$$

From (i) we have $y = 17 - 3x \quad \dots \quad (iii)$

„ (ii) „ $y = \frac{37 - 8x}{11} \quad \dots \quad (iv)$

Equating the values of y from (iii) and (iv), we have

$$17 - 3x = \frac{37 - 8x}{11}$$

or $187 - 33x = 37 - 8x$

or $-25x = -150$

or $x = 6.$

Substituting the value of x in (i), we have

$$3 \times 6 + y = 17$$

or $y = -1.$

Thus we have $x = 6$ and $y = -1.$

This is the **first method**. It consists in expressing one variable in terms of the other in both the equations and equating the results.

EXERCISE 51.

Solve the following equations by the above method :

1. $\begin{cases} x - 2y = 2 \\ 5x + 5y = 4. \end{cases}$

2. $\begin{cases} 4x - y = 6 \\ 3x - 2y = 7. \end{cases}$

3. $\begin{cases} x + 5y = 13 \\ 2x - 3y = 0. \end{cases}$

4. $\begin{cases} 3x - 4y = 0 \\ 2x + 3y = 17. \end{cases}$

5. $\begin{cases} 2x + 5y = 7 \\ 8y + 11x = 19. \end{cases}$

6. $\begin{cases} 3x + 4y = 19 \\ 6x - 5y = 25. \end{cases}$

7. $\begin{cases} 5x + 3y = 47 \\ 3x - 5y = 1. \end{cases}$

8. $\begin{cases} 3x - 2y = 10 \\ 2x + 4y = 12. \end{cases}$

$$9. \quad \begin{cases} x + ay = b \\ ax - by = c. \end{cases}$$

$$10. \quad \begin{cases} \frac{1}{3}(x - y) = \frac{1}{4}(y - 1) \\ \frac{1}{7}(4x - 5y) = x - 7. \end{cases}$$

Example 2. Solve the equations

$$5x + y = 14 \quad \dots \quad \dots \quad (i)$$

$$15x + 6y = 54 \quad \dots \quad \dots \quad (ii)$$

From (i) $y = -5x + 14$.

Now substituting this value of y in (ii), we have

$$15x + 6(-5x + 14) = 54 \quad \dots \quad (iii)$$

$$\text{or } 15x - 30x + 84 = 54$$

$$\text{or } -15x = -30$$

$$\text{or } x = 2.$$

To find y , substitute this value of x in (i)

$$\therefore 5 \times 2 + y = 14$$

$$\therefore y = 4.$$

$$\therefore \begin{cases} x = 2 \\ y = 4 \end{cases} \text{ is the required solution.}$$

This is the **second method**. It consists in expressing one variable in terms of the other in one equation only and substituting this value in the other equation.

Solve the following equations by the second method:

$$11. \quad \begin{cases} 3x + 2y = 9 \\ x + 3y = 10. \end{cases}$$

$$12. \quad \begin{cases} 7x - y = 5 \\ 2x + 3y = 8. \end{cases}$$

$$13. \quad \begin{cases} 4x + 7y = 62 \\ 3y - 2x = 8. \end{cases}$$

$$14. \quad \begin{cases} 2x + 5y + 7 = 0 \\ 3x + 7y + 10 = 0. \end{cases}$$

$$15. \quad \begin{cases} 7x + 4y = 5 \\ 5x + 6y = 2. \end{cases}$$

$$16. \quad \begin{cases} 3y = 7(x - 10) \\ 2(x - 5) = 3(y + 10). \end{cases}$$

$$17. \quad \begin{cases} 3x + 9y - 42 = 0 \\ 2x - 4y + 2 = 0. \end{cases}$$

$$18. \quad \left. \begin{aligned} \frac{7+x}{5} - \frac{2x-y}{4} &= 3y-5 \\ \frac{5y-7}{2} + \frac{4x-3}{6} &= 18-5x. \end{aligned} \right\}$$

$$19. \quad \begin{cases} \frac{1}{2}(3x - 2y) = \frac{1}{4}(2x - y) + 3 \\ \frac{1}{2}(5x - 4y) = \frac{1}{3}(4x - 3y) + 3. \end{cases}$$

$$20. \quad \begin{cases} ax + by = c \\ a^2x + b^2y = c^2. \end{cases}$$

Example 3. Solve the equations

$$3x + 4y = 20 \quad \dots \quad (i)$$

$$5x - 3y = 14 \quad \dots \quad (ii)$$

Multiplying (i) by 3 and (ii) by 4, we have

$$9x + 12y = 60 \quad \dots \quad (iii)$$

$$20x - 12y = 56 \quad \dots \quad (iv)$$

Adding (iii) and (iv), we have

$$29x = 116$$

$$x = 4.$$

Substituting this value of x in (i), we have

$$3 \times 4 + 4y = 20$$

$$\text{or} \quad 4y = 8$$

$$\text{or} \quad y = 2.$$

$$\therefore \quad \left. \begin{array}{l} x = 4 \\ y = 2 \end{array} \right\} \text{ is the required solution.}$$

This is the **third method**. It consists in equalising the co-efficients of one and the same variable in both the equations, then by subtraction or addition deriving an equation involving the other variable only.

This method is more frequently used than the other two :

NOTE. It is important to note that in each of these methods it is essential to *eliminate* or get rid of one of the two variables and thus to derive an equation involving a *single* variable.

Solve the following equations by the third method :

$$21. \quad \left. \begin{array}{l} x + y = 14 \\ x - y = 6. \end{array} \right\}$$

$$22. \quad \left. \begin{array}{l} x - y = 12 \\ x + y = 26. \end{array} \right\}$$

$$23. \quad \left. \begin{array}{l} 2x + 3y = 28 \\ 3x - 4y = -9. \end{array} \right\}$$

$$24. \quad \left. \begin{array}{l} 4x - 2y = 12 \\ 5x + 3y = 26. \end{array} \right\}$$

$$25. \quad \left. \begin{array}{l} 6x + 7y = 32 \\ 9x + y = 29. \end{array} \right\}$$

$$26. \quad \left. \begin{array}{l} 7x - 8y = -22 \\ 11x - 10y = 4. \end{array} \right\}$$

$$27. \quad \left. \begin{array}{l} 32x + 15y = 62 \\ 18x - 13y = -8. \end{array} \right\}$$

$$28. \quad \left. \begin{array}{l} 13x - 11y = 32 \\ 15x + 7y = 96. \end{array} \right\}$$

$$29. \quad \left. \begin{aligned} 2x + \cdot 4y &= 1\cdot 2 \\ 3\cdot 4x - \cdot 02y &= \cdot 01. \end{aligned} \right\}$$

$$30. \quad \left. \begin{aligned} \frac{1}{6}x + \frac{1}{16}y &= 6 \\ \frac{1}{12}y - \frac{1}{9}x &= 2 \end{aligned} \right\}$$

$$31. \quad 7x - 3y = 5x + y = 22.$$

$$32. \quad 14x + 3y = 6x + 15y = 48.$$

$$33. \quad 6x + 5y = 7x + 3y + 1 = 2(x + 6y - 1).$$

$$34. \quad \left. \begin{aligned} \frac{x-y}{3} &= \frac{y-1}{4} \\ \frac{4x-5y}{7} &= x-7. \end{aligned} \right\}$$

$$35. \quad \left. \begin{aligned} \frac{32+x}{5} - \frac{2x-y}{4} &= 3y \\ \frac{5y-43}{2} + \frac{4x-3}{6} &= -5x \end{aligned} \right\}$$

$$36. \quad \frac{x}{4} + \frac{y}{5} = \frac{x}{5} + \frac{y}{4} - 1 = 22.$$

$$37. \quad 3x - 2y + 2 = 5x - 3y + 1\frac{5}{8} = 6x - y - 4\frac{1}{2}.$$

$$38. \quad 7(x-5) = y-2, \quad \frac{4y-3}{2} = \frac{x+10}{6}.$$

$$39. \quad 7(2y-3x) - 2 = 2(9x-y), \quad 4x - 2y = 2(x+3) - 12.$$

$$40. \quad ax + by = c, \quad (a-b)x + (a-c)y = d.$$

Simultaneous equations involving the reciprocals of the variables may be solved as they stand, without clearing them of the fractions.

Example 4. Solve the equations

$$\frac{2}{x} + \frac{3}{y} = 2 \quad \dots \quad \dots \quad (i)$$

$$\frac{5}{x} + \frac{2}{y} = 3\frac{1}{6} \quad \dots \quad \dots \quad (ii)$$

Multiplying (i) by 2 and (ii) by 3, we have

$$\frac{4}{x} + \frac{6}{y} = 4 \quad \dots \quad \dots \quad (iii)$$

$$\frac{15}{x} + \frac{6}{y} = \frac{19}{2} \quad \dots \quad \dots \quad (iv)$$

Subtracting (iv) from (iii), we have

$$-\frac{11}{x} = -\frac{11}{2} \text{ and } x = 2.$$

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Substituting this value of x in (i), we have

$$\frac{2}{2} + \frac{3}{y} = 2$$

or $\frac{3}{y} = 1$ and $y = 3$.

$\therefore \begin{cases} x = 2 \\ y = 3 \end{cases}$ is the required solution.

Solve the equations:

$$41. \quad \left. \begin{aligned} 2x + \frac{3}{y} &= 5 \\ 5x - \frac{2}{y} &= 3. \end{aligned} \right\}$$

$$42. \quad \left. \begin{aligned} \frac{2}{x} + \frac{1}{y} &= \frac{11}{12} \\ \frac{1}{x} + \frac{3}{y} &= 1\frac{1}{12}. \end{aligned} \right\}$$

$$43. \quad \left. \begin{aligned} \frac{2}{x} + \frac{5}{y} &= 1 \\ \frac{3}{x} + \frac{2}{y} &= \frac{19}{20}. \end{aligned} \right\}$$

$$44. \quad \left. \begin{aligned} \frac{6}{x} + \frac{4}{y} &= 3 \\ \frac{9}{x} - \frac{1}{y} &= 2\frac{3}{4}. \end{aligned} \right\}$$

$$45. \quad \left. \begin{aligned} \frac{1}{3x} - \frac{1}{7y} &= \frac{2}{3} \\ \frac{1}{2x} - \frac{1}{3y} &= \frac{1}{6}. \end{aligned} \right\}$$

$$46. \quad \left. \begin{aligned} \frac{2}{3x} - \frac{5}{4y} &= \frac{1}{12} \\ \frac{3}{4x} - \frac{7}{6y} &= \frac{1}{3}. \end{aligned} \right\}$$

$$47. \quad \frac{m}{x} - \frac{n}{y} = a, \quad px = qy.$$

When the co-efficients of x and y are interchanged in two simultaneous equations, it is useful to employ the method of **addition** and **subtraction**, as illustrated below:

Example 5. Solve the equations

$$4x + 7y = 49 \quad \dots \quad \dots \quad (i)$$

$$7x + 4y = 61 \quad \dots \quad \dots \quad (ii)$$

Adding (i) and (ii), we have

$$11x + 11y = 110$$

or $x + y = 10 \quad \dots \quad \dots \quad (iii)$

Subtracting (i) from (ii), we have

$$3x - 3y = 12$$

or $x - y = 4 \quad \dots \quad \dots \quad (iv)$

Adding (iii) and (iv) and subtracting (iv) from (iii), we have

$$x=7 \text{ and } y=3.$$

Solve the equations:

$$48. \quad \left. \begin{array}{l} 10x + 11y = 20 \\ 11x + 10y = 22. \end{array} \right\}$$

$$50. \quad \left. \begin{array}{l} 11x + 3y = -21 \\ 3x + 11y = 35. \end{array} \right\}$$

$$52. \quad \left. \begin{array}{l} 3x + 20 = 4y - 10 \\ 4(x - 1) = 3(y - 3). \end{array} \right\}$$

$$54. \quad \left. \begin{array}{l} ax + by = m \\ bx + ay = n. \end{array} \right\}$$

$$49. \quad \left. \begin{array}{l} 9x + 5y = 60 \\ 5x + 9y = 52. \end{array} \right\}$$

$$51. \quad \left. \begin{array}{l} 17y + 6x = 28 \\ 17x + 6y + 5 = 0. \end{array} \right\}$$

$$53. \quad \left. \begin{array}{l} \frac{1}{6}x + \frac{1}{4}y = \frac{13}{6} \\ \frac{1}{4}x + \frac{1}{6}y = 2. \end{array} \right\}$$

$$55. \quad \left. \begin{array}{l} 49x - 57y = 172 \\ 57x - 49y = 252. \end{array} \right\}$$

2. Simultaneous equations involving three unknown quantities.

Example 1. Solve $3x + y + 2z = 13$... (i)

$$x + 4y + 3z = 14 \quad \dots \quad \dots \quad \text{(ii)}$$

$$2x + 3y + 4z = 16 \quad \dots \quad \dots \quad \text{(iii)}$$

Multiplying (i) by 3 and (ii) by 2, we have

$$9x + 3y + 6z = 39 \quad \dots \quad \dots \quad \text{(iv)}$$

$$2x + 8y + 6z = 28 \quad \dots \quad \dots \quad \text{(v)}$$

Subtracting (v) from (iv), we have

$$7x - 5y = 11 \quad \dots \quad \dots \quad \text{(vi)}$$

Again multiplying (ii) by 4 and (iii) by 3, we have

$$4x + 16y + 12z = 56 \quad \dots \quad \dots \quad \text{(vii)}$$

$$6x + 9y + 12z = 48 \quad \dots \quad \dots \quad \text{(viii)}$$

Subtracting (viii) from (vii), we have

$$-2x + 7y = 8 \quad \dots \quad \dots \quad \text{(ix)}$$

Multiplying (vi) by 7 and (ix) by 5, we have

$$49x - 35y = 77 \quad \dots \quad \dots \quad \text{(x)}$$

$$-10x + 35y = 40 \quad \dots \quad \dots \quad \text{(xi)}$$

Adding (x) and (xi), we have

$$39x = 117$$

$$\text{or } x = 3.$$

Substituting this value in (vi), we have

$$21 - 5y = 11, \quad \therefore y = 2.$$

Substituting the values of x and y in (i), we have

$$9 + 2 + 2z = 13, \therefore z = 1.$$

Thus

$$x = 3, y = 2 \text{ and } z = 1.$$

Example 2. Solve $y + z = 17$... (i)

$$z + x = 14 \quad \dots \quad \dots \quad \text{(ii)}$$

$$x + y = 11 \quad \dots \quad \dots \quad \text{(iii)}$$

Adding (i), (ii) and (iii), we have

$$2(x + y + z) = 42$$

$$x + y + z = 21. \quad \dots \quad \dots \quad \text{(iv)}$$

Subtracting (i), (ii) and (iii), respectively from (iv), we have

$$x = 4, y = 7, z = 10.$$

Example 3. Solve $\frac{xy}{x+y} = \frac{1}{3}$... (i)

$$\frac{yz}{y+z} = \frac{1}{5} \quad \dots \quad \dots \quad \text{(ii)}$$

$$\frac{zx}{z+x} = \frac{1}{4} \quad \dots \quad \dots \quad \text{(iii)}$$

Taking the reciprocals of (i), (ii) and (iii), we have

$$\frac{x+y}{xy} = 3, \text{ or } \frac{1}{y} + \frac{1}{x} = 3 \quad \dots \quad \dots \quad \text{(iv)}$$

$$\frac{y+z}{yz} = 5, \text{ or } \frac{1}{z} + \frac{1}{y} = 5 \quad \dots \quad \dots \quad \text{(v)}$$

$$\frac{z+x}{zx} = 4, \text{ or } \frac{1}{x} + \frac{1}{z} = 4 \quad \dots \quad \dots \quad \text{(vi)}$$

Adding (iv), (v) and (vi), we get

$$2\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 12$$

$$\text{or } \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 6 \quad \dots \quad \dots \quad \text{(vii)}$$

Subtracting (iv), (v) and (vi) respectively from (vii),

$$\left. \begin{aligned} \text{we have } \frac{1}{z} &= 3, \therefore z = \frac{1}{3} \\ \frac{1}{x} &= 1, \therefore x = 1 \\ \frac{1}{y} &= 2, \therefore y = \frac{1}{2} \end{aligned} \right\}$$

Example 4. Solve $x + ay + a^2z = a^3$... (i)
 $x + by + b^2z = b^3$... (ii)
 $x + cy + c^2z = c^3$... (iii)

Subtracting (ii) from (i), we have

$$y(a-b) + z(a^2 - b^2) = a^3 - b^3$$

or $y + z(a+b) = a^2 + ab + b^2$... (iv)

Subtracting (iii) from (ii), we have

$$y(b-c) + z(b^2 - c^2) = b^3 - c^3$$

or $y + z(b+c) = b^2 + bc + c^2$... (v)

Subtracting (v) from (iv), we have

$$z(a-c) = (a^2 - c^2) + b(a-c)$$

or $z = a + c + b$
 $= a + b + c.$

Substituting the value of z in (iv), we have

$$y + (a+b+c)(a+b) = a^2 + ab + b^2$$

or $y = -(ab + bc + ca).$

Substituting the values of y and z in (i), we have

$$x - a(ab + bc + ca) + a^2(a + b + c) = a^3$$

or $x = abc.$

EXERCISE 52.

Solve the following equations :

$$\left. \begin{array}{l} 1. \quad x - y + z = 1 \\ \quad x - 2y + 4z = 8 \\ \quad x - 3y + 9z = 27. \end{array} \right\}$$

$$\left. \begin{array}{l} 2. \quad x + 2y + 3z = 14 \\ \quad 2x + 3y + 4z = 20 \\ \quad 3x + y + 6z = 23. \end{array} \right\}$$

$$\left. \begin{array}{l} 3. \quad x - 3y + 2z = 1 \\ \quad 2x - 4y + z = 4 \\ \quad 3x + y - 5z = -11. \end{array} \right\}$$

$$\left. \begin{array}{l} 4. \quad x - y - z = -2 \\ \quad x + y + 2z = 7 \\ \quad 5x + 6y + 4z = 32. \end{array} \right\}$$

$$\left. \begin{array}{l} 5. \quad x + 2y + 3z = 22 \\ \quad 2x - 3y + z = 5 \\ \quad 3x + 4y - 2z = 7. \end{array} \right\}$$

$$\left. \begin{array}{l} 6. \quad x + 2y + 3z = 20 \\ \quad 2x + 3y - 5z = -7 \\ \quad 4x - 5y + 7z = 21. \end{array} \right\}$$

$$\left. \begin{aligned} 7. \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} &= 6 \\ \frac{2}{x} - \frac{3}{y} + \frac{4}{z} &= 8 \\ \frac{3}{x} - \frac{4}{y} + \frac{5}{z} &= 10 \end{aligned} \right\}$$

$$\left. \begin{aligned} 8. \quad \frac{1}{x} + \frac{1}{y} - \frac{1}{z} &= 2 \\ \frac{3}{x} - \frac{4}{y} + \frac{5}{z} &= 18 \\ \frac{2}{x} + \frac{3}{y} + \frac{4}{z} &= 38 \end{aligned} \right\}$$

9. $x + y = 20, y + z = 16, z + x = 18.$

10. $x + y = 2a, y + z = 2b, z + x = 2c.$

11. $\frac{1}{x} + \frac{1}{y} = 1, \frac{1}{y} + \frac{1}{z} = 2, \frac{1}{z} + \frac{1}{x} = 4.$

12. $\frac{1}{x} + \frac{1}{y} = a, \frac{1}{y} + \frac{1}{z} = b, \frac{1}{z} + \frac{1}{x} = c.$

$$\left. \begin{aligned} 13. \quad y + z - x &= 8 \\ z + x - y &= 12 \\ x + y - z &= 16. \end{aligned} \right\}$$

$$\left. \begin{aligned} 14. \quad x - 3y + z &= -3 \\ y - 3z + x &= 1 \\ z - 3x + y &= -7. \end{aligned} \right\}$$

15. $ay + bx = c, cx + az = b, bz + cy = a.$

[Hint. Multiply first by c , second by b and third by a .]

16. $\frac{x+y}{xy} = \frac{y+z}{yz} = \frac{z+x}{zx} = \frac{2}{5}.$

$$\left. \begin{aligned} 17. \quad 2x + 3y &= 5xy \\ 3y + 4z &= 7yz \\ 4z + 5x &= 9zx. \end{aligned} \right\}$$

$$\left. \begin{aligned} 18. \quad 3xy &= 4(x+y) \\ 2xz &= 3(x+z) \\ 5yz &= 12(y+z). \end{aligned} \right\}$$

$$\left. \begin{aligned} 19. \quad x + y &= axy \\ y + z &= byz \\ z + x &= czx. \end{aligned} \right\}$$

$$\left. \begin{aligned} 20. \quad x - ay + a^2z &= a^3 \\ x - by + b^2z &= b^3 \\ x - cy + c^2z &= c^3. \end{aligned} \right\}$$

$$\left. \begin{aligned} 21. \quad x + ay + bcz &= a^2 \\ x + by + caz &= b^2 \\ x + cy + abz &= c^2. \end{aligned} \right\}$$

22. $xy = 20, yz = 24, zx = 30.$
[Hint. Multiply the three equations and take the square root.]

23. $\left. \begin{aligned} x(x+y+z) &= 60 \\ y(x+y+z) &= 75 \\ z(x+y+z) &= 90. \end{aligned} \right\}$ [Hint. Add the three equations.]

$$\left. \begin{aligned} 24. \quad (x+y)(y+z) &= 96 \\ (y+z)(z+x) &= 80 \\ (z+x)(x+y) &= 120. \end{aligned} \right\}$$

Example 5. Solve $x+y=10$

$$xy=24$$

$$(x-y)^2 = (x+y)^2 - 4xy$$

or

$$x-y = \sqrt{(x+y)^2 - 4xy}$$

$$= \sqrt{10^2 - 4 \cdot 24}$$

$$= \sqrt{100 - 96} = \sqrt{4} = 2. \quad \dots \quad (iii)$$

Adding (i) and (iii), we have

$$2x = 12, \quad \therefore x = 6.$$

Subtracting (iii) from (i), we have

$$2y = 8, \quad \therefore y = 4.$$

Thus $x=6$ and $y=4$.

Solve the equations:

$$25. \quad \left. \begin{array}{l} x+y=11 \\ xy=28. \end{array} \right\}$$

$$26. \quad \left. \begin{array}{l} x+y=17 \\ xy=60. \end{array} \right\}$$

$$27. \quad \left. \begin{array}{l} x-y=7 \\ xy=120. \end{array} \right\}$$

$$28. \quad \left. \begin{array}{l} x-y=5 \\ xy=84. \end{array} \right\}$$

$$29. \quad \left. \begin{array}{l} x+y=15 \\ xy=56. \end{array} \right\}$$

$$30. \quad \left. \begin{array}{l} x-y=4 \\ xy=77. \end{array} \right\}$$

$$31. \quad \left. \begin{array}{l} x+y=a \\ xy=b^2. \end{array} \right\}$$

$$32. \quad \left. \begin{array}{l} x-y=m \\ xy=n^2. \end{array} \right\}$$

$$33. \quad \left. \begin{array}{l} \frac{1}{x} + \frac{1}{y} = \frac{7}{12} \\ \frac{1}{xy} = \frac{1}{12}. \end{array} \right\}$$

$$34. \quad \left. \begin{array}{l} \frac{1}{x} - \frac{1}{y} = \frac{3}{10} \\ \frac{1}{xy} = \frac{1}{10}. \end{array} \right\}$$

3. Problems involving simultaneous equations.

Example 1. Find two numbers such that four times the first added to three times the second is 93 and the excess of three times the first over twice the second is 6.

Let x be the first number, and y the second.

Four times the first + three times the second = $4x + 3y$,

\therefore the first equation is $4x + 3y = 93 \quad \dots \quad (i)$

The excess of three times the first over twice the second = $3x - 2y$,

$$\text{and } \frac{x}{y+8} = \frac{1}{4}$$

$$\text{or } 4x - y = 8 \quad \dots \quad (ii)$$

Subtracting (i) from (ii), we have

$$x = 5.$$

Substituting this value of x in (i), we have

$$15 - y = 3$$

$$\text{or } y = 12.$$

Hence the required fraction is $\frac{5}{12}$.

Example 4. A number consists of three digits whose sum is 17; the middle digit exceeds the sum of the other two by 1; if the digits be reversed, the number is diminished by 396. Find the number.

Let x , y and z be the digits in hundreds, tens and units place respectively.

Since the sum of the digits $= x + y + z$,

$$\therefore x + y + z = 17 \quad \dots \quad (i)$$

Since the middle digit y exceeds the sum of the other digits x and z by 1,

$$\therefore y = x + z + 1$$

$$\text{or } x - y + z = -1 \quad \dots \quad (ii)$$

Since the actual number is $100x + 10y + z$ and it becomes $100z + 10y + x$ when the digits are reversed,

$$\therefore (100x + 10y + z) - (100z + 10y + x) = 396.$$

Simplifying and dividing both sides by 99, we have

$$x - z = 4 \quad \dots \quad (iii)$$

Adding (i) and (ii), we have

$$2x + 2z = 16$$

$$\text{or } x + z = 8 \quad \dots \quad (iv)$$

From equations (iii) and (iv), we have

$$x = 6 \text{ and } z = 2.$$

Substituting the values of x and z in (i), we have

$$6 + y + 2 = 17, \quad \therefore y = 9.$$

Hence the required number is 692.

EXERCISE 53.

1. Find two numbers whose sum is 39 and difference is 7.
2. Find two numbers such that three times their sum is 84 and twice their difference is 4.
3. Find two numbers such that four times the first added to the second is equal to 96 and the excess of three times the first over the second is equal to 51.
4. Find two numbers such that three times the first added to four times the second is equal to 114 and four times the first added to three times the second is equal to 117.
5. Two decimal fractions have their sum equal to $\cdot 56$ and three times their difference equal to $\cdot 24$. Find them.
6. A straight line $27''$ in length is divided into two parts such that six times the first part exceeds four times the other by $12''$. Find each part.
7. The sum of three numbers is 77. Three times the first added to the third is less than four times the second by 3. The excess of the third over the second is 18. Find the numbers.
8. The sum of two numbers is increased by 10 and the result divided by 8; the quotient is less than their difference by 1. Also $\frac{1}{2}$ of the greater is greater than $\frac{1}{4}$ of the less by 5. Find the two numbers.
9. Find three numbers such that the excess of the first over the second is 8, the excess of the second over the third is 4, and the sum of the first and the third is 28.
10. A and B have Rs. 320 between them. A gains from B $\frac{1}{5}$ of B 's money. Then B gains from A Rs. 16. Each has now the same amount. Find the original sum of each.
11. Find three numbers x, y, z such that x increased by $\frac{1}{2}$ of y , y increased by $\frac{1}{3}$ of z , and z increased by $\frac{1}{4}$ of x may each be 250.
12. 25 horses and 12 cows together cost Rs. 2,350; also 18 horses and 15 cows cost Rs. 2,010. Find the cost price of a horse and a cow.

13. 16 maunds of wheat and 6 maunds of rice together cost Rs. 89, and 24 maunds of wheat and 15 maunds of rice cost Rs. 166 8 as. Find the rates of wheat and rice per maund.

14. The price of 12 tables and 20 chairs is Rs. 164 and the price of 15 tables and 24 chairs is Rs. 201. Find the price of a table and a chair.

15. A certain fraction becomes $\frac{2}{3}$ when its numerator is diminished by 1, and becomes $\frac{1}{2}$ when its denominator is increased by 5. Find the fraction.

16. What fraction is that which becomes 1 if 1 be added to its numerator and becomes $\frac{1}{2}$ if 1 be added to its denominator?

17. What fraction is that which becomes $\frac{2}{3}$ if its numerator be doubled and denominator increased by 9 and becomes $\frac{1}{2}$ if its numerator be increased by 7 and denominator doubled?

18. If the denominator of a fraction be added to the numerator and the numerator be subtracted from the denominator, it becomes $2\frac{2}{3}$. The denominator of the fraction exceeds twice its numerator by 1. Find the fraction.

19. The denominator of a fraction exceeds the numerator by 4, and if 5 be taken away from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5. Find the original fraction. [B. U. 1892.]

20. A person bought a horse and a carriage for Rs. 810; if the price of the horse were 10% more and that of the carriage 10% less, he would have to pay Rs. 799. Find the cost price of the horse and also that of the carriage.

21. A person sells two articles together for Rs. 46, making 10% profit on one and 20% profit on the other. If he had sold each article at 15% profit, the result would have been the same. At what price does he sell each article?

(C. U. 1891.)

22. A person has two horses and a saddle worth Rs. 95; if the saddle be put on the first horse, his value becomes

double that of the second ; but if the saddle be put on the second horse, his value would be less than that of the first by Rs. 125. What is the value of each ?

23. A number of two digits when divided by 5 gives a certain quotient and a remainder 4 ; when divided by 8, gives another quotient and a remainder 7. The digit in the tens place is equal to the second quotient diminished by 2 and the other digit is less than the first quotient by 6. Find the number.

24. A man had a sum of 16s. 6d. in shillings and pence. If the number of shillings and pence were interchanged, he would gain 2s. 9d. Find how many shillings and pence he had.

25. The cost of 12 lbs. of tea and 15 lbs. of butter is £3 19s. 6d. ; the cost of 8 lbs. of tea and 6 lbs. of butter is £2 3s. Find the cost of each per lb.

26. A number consists of two digits. When it is divided by the sum of the digits, the quotient is 7. The sum of the reciprocals of the digits is 9 times the reciprocal of the product of the digits. Find the number. (M. M. 1887.)

27. Reverse the digits of a number, it becomes $\frac{5}{8}$ of what it was before ; also the difference between the two digits is one. Find the number. (C. E. 1883.)

28. A certain number of 3 digits exceeds the sum of its digits by 414 and the number formed by reversing the digits exceeds the same sum by 612. If the given number and one formed by reversing the digits be in the ratio of 71 : 104, find the number.

29. A number consists of 3 digits whose sum is 18. The digit in the hundreds place is $\frac{1}{8}$ of the number formed by the remaining two digits, and the digit in the units place is $\frac{1}{8}$ of the number formed by the remaining two digits. Find the number.

30. A number consists of 3 digits whose sum is 10. The middle digit is equal to the sum of the other two. The number formed by reversing the digits is greater than the original number by 99. Find the number.

Example 5. Two persons A and B , 20 miles apart, starting at the same time, meet in $2\frac{1}{2}$ hours if they walk in opposite directions; but they meet in 10 hours, if they start at the same time and walk in the same direction. Find their speed per hour.

Let x miles per hour be the speed of A and y miles per hour the speed of B .

When they walk in opposite directions, their speed of approach is $(x+y)$ miles per hour and since they together cover 20 miles in $2\frac{1}{2}$ hours,

$$\begin{aligned} \therefore 2\frac{1}{2}(x+y) &= 20 \\ \text{or } x+y &= 8 \dots\dots\dots (i) \end{aligned}$$

When they walk in the same direction, their speed of approach is $(x-y)$ miles per hour. Since A overtakes B after making up a distance of 20 miles in 10 hours,

$$\begin{aligned} \therefore 10(x-y) &= 20 \\ \text{or } x-y &= 2 \dots\dots\dots (ii) \end{aligned}$$

From (i) and (ii), we have

$$x = 5 \text{ and } y = 3.$$

Hence the speed of A is 5 miles an hour and the speed of B is 3 miles an hour.

Example 6. A sum of money is divided equally among a certain number of men; had there been eight more, each would have received Re. 1 less, and had there been four fewer, each would have received Re. 1 more than he did. Find the number of men and the sum of money.

Let x be the number of men and y the number of rupees each gets.

Then the sum divided $= xy$ rupees.

$$\begin{aligned} \text{By the question } (x+8)(y-1) &= xy \dots\dots\dots (i) \\ \text{and } (x-4)(y+1) &= xy \dots\dots\dots (ii) \end{aligned}$$

From (i), we have

$$\begin{aligned} xy - x + 8y - 8 &= xy \\ \therefore -x + 8y - 8 &= 0 \dots\dots\dots (iii) \end{aligned}$$

From (ii), we have

$$\begin{aligned} xy + x - 4y - 4 &= xy \\ x - 4y - 4 &= 0 \dots\dots\dots (iv) \end{aligned}$$

Adding (iii) and (iv), we have

$$4y - 12 = 0, \quad \therefore y = 3.$$

Substituting this value of y in (iv), we have

$$x - 12 - 4 = 0, \quad \therefore x = 16.$$

Hence the number of men = 16 and the sum divided = Rs. 3×16 = Rs. 48.

***Example 7.** Two persons have together 5 mds. of luggage and are charged for the excess above the ordinary concession Rs. 5 and Rs. 3 respectively. If all the luggage had belonged to one man, he would have been charged Rs. 9. How much luggage had each passenger and what was the limit of concession per passenger?

Let x mds. be the luggage of the first passenger, then $(5 - x)$ mds. would be the luggage of the second passenger.

Let y mds. of luggage be allowed to each to carry free of charge and let z rupees per md. be the rate of charge for the excess luggage.

Then the excess luggage carried by the first passenger is $(x - y)$ mds. and the charge on it = Rs. $(x - y)z$,

$$\therefore (x - y)z = 5 \quad \dots \dots \dots (i)$$

The excess luggage carried by the second passenger is $(5 - x - y)$ mds. and the charge on it = Rs. $(5 - x - y)z$,

$$\therefore (5 - x - y)z = 3 \quad \dots \dots \dots (ii)$$

If all the luggage had belonged to one man, the excess luggage would have been $(5 - y)$ mds. and the charge on it would have been $(5 - y)z$ rupees.

$$\therefore (5 - y)z = 9 \quad \dots \dots \dots (iii)$$

Dividing (ii) by (i), we have

$$\frac{5 - x - y}{x - y} = \frac{3}{5}$$

$$\text{or} \quad 5(5 - x - y) = 3(x - y) \quad \dots \dots \dots (iv)$$

Dividing (iii) by (i), we have

$$\frac{5 - y}{x - y} = \frac{9}{5}$$

$$\text{or} \quad 5(5 - y) = 9(x - y) \quad \dots \dots \dots (v)$$

From (iv) and (v), we have

$$x = 3 \text{ and } y = \frac{1}{2}.$$

Substituting the values of x and y in (i), we have

$$(3 - \frac{1}{2})z = 5, \quad \therefore z = 2$$

Hence the luggage of one man = 3 mds., of the second = 2 mds., and the concession allowed per passenger = $\frac{1}{2}$ md.

31. A person goes a certain distance on bicycle. If he had gone 3 miles an hour faster, he would have taken $\frac{2}{3}$ hour less; but if he had gone 2 miles an hour slower, he would have taken $\frac{2}{3}$ hour longer. Find the distance.

32. A person walks on the first day a certain distance at the rate of 4 miles an hour and on the second day a certain distance at the rate of 3 miles an hour, and thus completes a journey of 30 miles. If he had walked the first distance at 3 miles an hour and the second distance at 4 miles an hour, he would have taken $\frac{1}{2}$ hour more. Find the time he took to walk the whole distance.

33. Two persons, 15 miles apart, start at the same time. If they walk in opposite directions, they meet in 2 hours; but if they walk in the same direction, they meet in 30 hours. Find their rates of walking.

34. A person rows down-stream 35 miles in 7 hours and up-stream the same distance in $17\frac{1}{2}$ hours. Find the rate at which he rows and the speed of the stream.

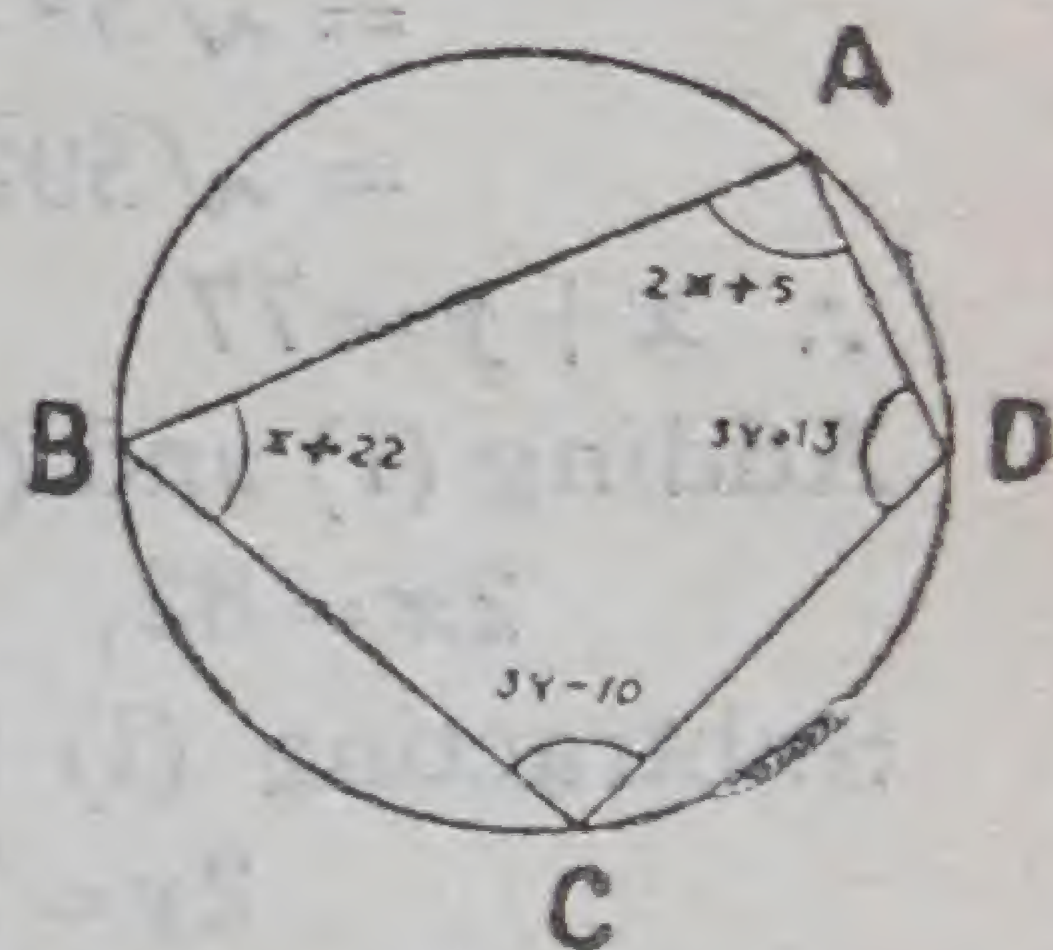
35. A messenger is allowed 16 hours for going from P to Q and back again, including 4 hours' rest at Q . But if he were to go 2 miles an hour faster each way, he would be able to take 5 hours' rest at Q . Find the speed and the distance from P to Q .

36. A person rows down-stream 26 miles and up-stream the same distance in $19\frac{1}{2}$ hours. Also the difference of time taken by him to row 39 miles up and down is 9 hours and 45 minutes. Find the rate at which he rows and the speed of the stream.

37. A boat goes up-stream 30 miles and down-stream 44 miles in 10 hours. It also goes up-stream 40 miles and down-stream 55 miles in 13 hours. Find the speeds of the boat and the stream.

38. The perimeter of a rectangle is 72 ft. If its length be decreased by 6 ft. and breadth increased by 4 ft., the area is unaltered. Find the area.

39. In a cyclic quadrilateral the sum of each pair of opposite angles is 180° . The number of degrees in its angles is given in the figure in x and y . Find x and y .



40. A merchant has two kinds of tea, one worth 2s. per lb. and the other worth 3s. 4d. per lb.; from these he prepares a mixture of 120 lbs. worth 2s. 4d. per lb. How much of each kind must he mix?

41. Three ladies buy 3 kinds of cloth, A , B and C . The first buys 5 yds. of A and 3 yds. of B , spending the same amount on each; the second buys 10 yds. of A and $5\frac{1}{2}$ yds. of C , spending £5; while the third buys 3 yds. of each kind of cloth for £3 6s. Find the price per yard of each kind of cloth.

*42. Two persons have together 6 mds. of luggage and are charged for the excess above the ordinary concession Re. 1 4 as. and Re 1 14 as. respectively. If all the luggage had belonged to one man, he would have been charged Rs. 3 7 as. How much luggage had each passenger and what is the limit of concession per passenger?

43. The monthly expenses of a family, when wheat is selling @ 18 srs. for a rupee, are Rs. 60; when it is selling @ 20 srs. for a rupee, expenses are Rs. 58, other expenses remaining unchanged. What will be the expenses when wheat is selling @ 24 srs. for a rupee?

***Example 8.** A rectangular garden is 45 yds. long and 40 yds. broad. A road of uniform width is constructed inside all round it. The area of the remaining garden is 1476 sq. yds. Find the width of the road.

Let the length of the inner rectangle be x yds. and its breadth y yds.

Obviously, $x - y = 45 - 40 = 5 \dots (i)$

and $xy = 1476 \dots (ii)$

$$\begin{aligned} \therefore x + y &= \sqrt{(x - y)^2 + 4xy} \\ &= \sqrt{5^2 + 4 \times 1476} \\ &= \sqrt{5929} = 77. \end{aligned}$$

$\therefore x + y = 77 \dots (iii)$

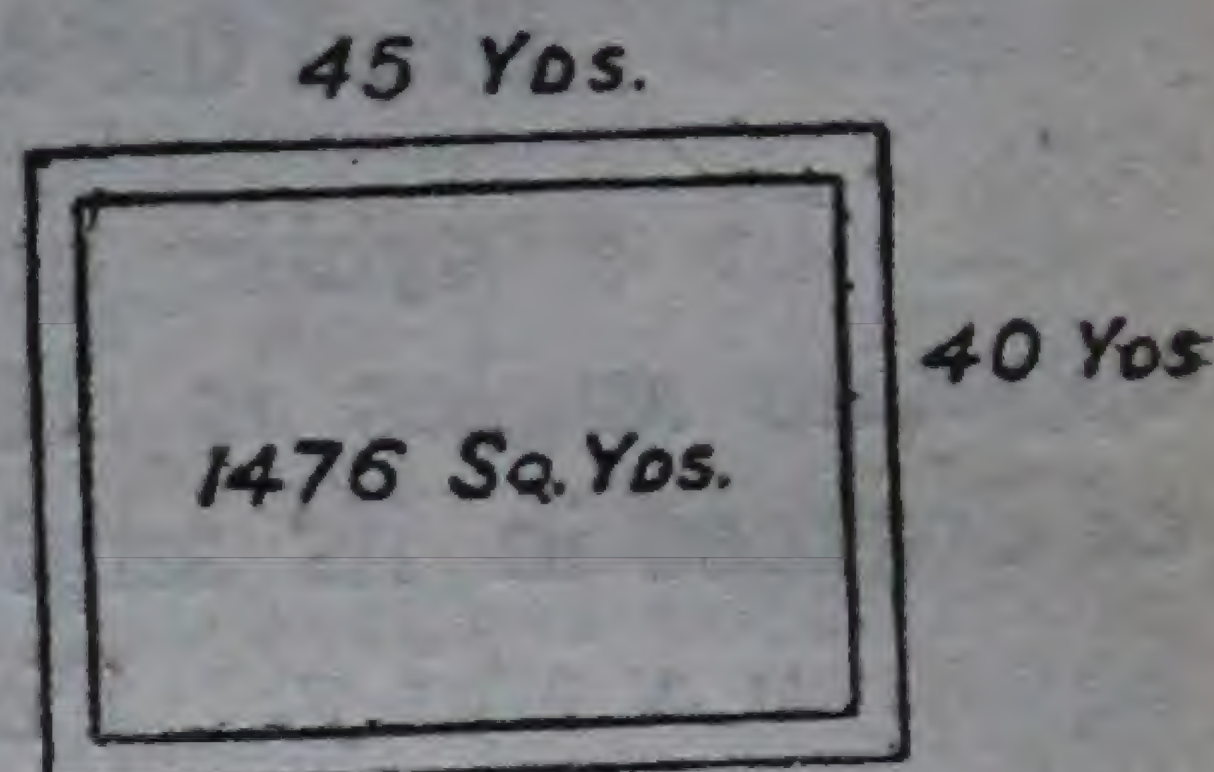
Adding (i) and (iii), we have

$$2x = 82, \quad \therefore x = 41.$$

Subtracting (i) from (iii), we have

$$2y = 72, \quad \therefore y = 36.$$

\therefore width of the road $= \frac{1}{2} (45 - 41) = 2$ yds.



NOTE. In examples of this type, when the path is of uniform width, the difference between the length and the breadth of the inner rectangle is the same as that of the outer rectangle.

***44.** The area of a rectangle is 1728 sq. ft. and its semi-perimeter is 84 ft. Find its length and breadth.

***45.** The area of a triangle is 80 sq. ft., the sum of its base and altitude is 28 ft. Find its base and altitude.

***46.** The area of a rectangle is 3840 sq. ft. and its length is greater than its breadth by 32 ft. Find its length and breadth.

***47.** The top of a rectangular table is 40" long and 28" wide. Leaving out a uniform margin all round, it is covered with oil cloth whose area is 640 sq. inches. Find the width of the margin.

***48.** The difference between the length and breadth of a rectangle is 14 ft. and its area is 275 sq. ft.

(i) Find its semi-perimeter, (ii) find its length and breadth. [P. U. 1931.]

SECTIONAL REVISION III

TEST PAPERS

PAPER 1

1. Shew that the H.C.F. of two expressions A and B divides exactly their sum as well as difference.

2. Simplify (i) $\frac{a}{a^2 - b^2} - \frac{b}{b^2 - a^2}$.

(ii) $\frac{a^2 - 3ab + 2b^2}{a - 2b} + \frac{6a^2 - 5ab - 6b^2}{2a - 3b} - \frac{6a^2 + ab - 2b^2}{3a + 2b}$.

3. Solve the equations
$$\left. \begin{aligned} 5x + \frac{3}{y} &= 20\frac{1}{2} \\ 3x - \frac{4}{y} &= 11\frac{1}{3} \end{aligned} \right\}$$

4. Find two numbers such that three times the first added to five times the second is equal to 53, and also such that three times the second subtracted from four times the first is equal to 3.

5. Find the value of $x^3 + \frac{1}{x^3}$ when $x + \frac{1}{x} = 4$.

PAPER 2

1. Find the H.C.F. of $6x^3 - 4x^2 + 3x - 2$, $6x^3 - 31x^2 + 30x - 8$ and $12x^3 - 2x^2 - 7x + 2$.

2. (i) Express $\frac{x^3 - 1}{x + a}$ in a form partly integral and partly fractional.

(ii) Simplify $\frac{x+1}{2x^3-4x^2} + \frac{x-1}{2x^3+4x^2} - \frac{1}{x^2-4}$.

3. Solve the equations
$$\left. \begin{aligned} \frac{12}{x} - \frac{5}{y} &= 3 \\ \frac{9}{x} + \frac{2}{y} &= 3\frac{2}{5} \end{aligned} \right\}$$

4. Eight years back A was twice as old as B . After 15 years their united ages will be 82. Find their present ages.

5. A man's monthly income is progressive and varies as below : Rs. 50, Rs. 56, Rs. 62, Rs. 68, Rs. 74, Rs. 80 &c. Find a formula which determines his monthly income in general.

PAPER 3

1. Find the H.C.F. of $2x^5 - 5x^2 + 3$ and $3x^5 - 5x^3 + 2$ by the method of alternate destruction of the highest and the lowest terms.

2. Simplify
$$\frac{5}{x^2 - 3x - 28} + \frac{3}{x^2 + x - 12} + \frac{9}{x^2 - 10x + 21}.$$

3. Solve the equations
$$\left. \begin{aligned} 13x + 9y &= 57 \\ 9x + 13y &= 53 \end{aligned} \right\}$$

4. A number of two digits is reversed if it is diminished by 9; the sum of its digits is 15. Find the number.

5. Shew that $(4x^2 - 2x - 15)^2 - (2x^2 - 5x - 9)^2$ is divisible by $2x^2 + 3x - 6$ and express the quotient in the form of factors.

PAPER 4

1. Reduce to the lowest terms
$$\frac{6x^4 + 25x^3 + 25x^2 + 13x + 3}{3x^4 + 11x^3 + 13x^2 + 7x + 2}.$$

2. Simplify
$$\frac{1+x^2}{1-x^2} + \frac{4x^2}{1+x^4} - \frac{1-x^2}{1+x^2}.$$

3. Solve the equations
$$\left. \begin{aligned} \frac{1}{x} + \frac{1}{y} &= \frac{9}{20} \\ \frac{1}{y} + \frac{1}{z} &= \frac{8}{15} \\ \frac{1}{z} + \frac{1}{x} &= \frac{7}{12} \end{aligned} \right\}$$

4. A person does a journey in a motor car at a uniform speed in 6 hours. On his return he stops at the half-way for half an hour, but quickening his speed by 3 miles an hour completes it in the same time. Find his original speed and the length of the journey.

5. Find the value of $a^3 + b^3 + c^3 - 3abc$ when $a + b + c = 9$ and $a^2 + b^2 + c^2 = 29$.

PAPER 5

1. If A and B are two algebraic expressions and $A = a.H$, $B = b.H$. prove that the L.C.M. $= \frac{A}{H} \cdot B$.

2. Simplify $\frac{6x^2 - 5xy - 6y^2}{14x^2 - 23xy + 3y^2} - \frac{15x^2 + 8xy - 12y^2}{35x^2 + 47xy + 6y^2}$.

3. Solve the equations
$$\left. \begin{aligned} 2x + 3y + z &= 19 \\ x + 2y - z &= 8 \\ 3x - y + 2z &= 13 \end{aligned} \right\}$$

4. A person goes on a bicycle from A to B and back in a certain time at 12 miles an hour. If he had gone 10 miles an hour from A to B and returned at 16 miles an hour, he would have taken $6\frac{1}{4}$ minutes less for the double journey. Find the distance from A to B .

5. Factorise (i) $(x^2 - 3x)(x^2 - 3x - 2) - 24$.
(ii) $81x^4 + 64$.

PAPER 6

1. If A and B are two expressions, H and L their H.C.F. and L.C.M. respectively, prove that $A.B = HL$.

2. Simplify $\frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \frac{8}{x^8-1}$.

3. Solve the equations
$$\left. \begin{aligned} \frac{1}{2x} - \frac{1}{3y} &= \frac{1}{12} \\ \frac{1}{6xy} &= \frac{1}{72} \end{aligned} \right\}$$

4. Divide Rs. 1,440 into two parts such that if they are invested at $3\frac{1}{2}\%$ and 5% respectively, they may together yield the same income as if the whole were invested at $4\frac{1}{2}\%$.

5. Find the value of $x^2 - y^2 + 6x + 10y - 16$ when $x + y = 15$ and $x - y = 3$.

PAPER 7

1. The H.C.F. of two expressions is $x + 2y$ and their L.C.M. is $6x^3 + 7x^2y - 16xy^2 - 12y^3$, one expression is $2x^2 + xy - 6y^2$; find the other.

2. Simplify $\frac{3}{2x+3 - \frac{3}{1 - \frac{x}{x+6}}}$.

3. What is meant by an indeterminate equation? Illustrate it by an example.

4. A person buys 50 lbs. of tea and 60 lbs. of coffee for £10 5s. By selling the coffee at a loss of 5% and the tea at a gain of 10% , he makes a profit of 8s. 6d. What was the cost price of tea and coffee per lb.

5. Collect the co-efficient of x^2 from the product of $(2x^2 + 5x - 1)$ and $(3x^2 - 2x + 4)$ without actual multiplication.

PAPER 8

1. Find the H.C.F. and L.C.M. of $x^5 + x^3$, $x^6 - x^2$, $x^4 - 3x^3 + 3x^2 - 3x + 2$.

2. Simplify $x + 1 - \frac{x}{x+2 - \frac{x+1}{x + \frac{1}{x+2}}}$.

3. Solve the equations $\left. \begin{array}{l} 2x + 5y = 33 \\ 4x - 3y = 1 \end{array} \right\}$

by all the three methods.

4. A rectangular garden is 36 yds. long and 25 yds. broad. A road of uniform width is constructed inside all round it. The area of the remaining garden is 672 sq. yds. Find the width of the road.

5. Factorise $2a^3 - 7a^2 + 4a - 3$.

PAPER 9

1. Find the L. C. M. of $3x^3 + 17ax^2 - 62a^2x + 14a^3$ and $7x^3 + 52ax^2 - 46a^2x + 8a^3$.

2. Simplify $\left\{ 1 + \frac{2b^2}{a(a-3b)} \right\} \left\{ 1 + \frac{b}{2b-a} \right\} - \left(\frac{a^2}{b^2} + \frac{b}{a} \right) \left(\frac{a^2 - ab}{a^2 - ab + b^2} - 1 \right)$.

3. Solve the equations
$$\left. \begin{aligned} \frac{xy}{x+y} &= \frac{6}{5} \\ \frac{yz}{y+z} &= \frac{12}{7} \\ \frac{zx}{z+x} &= \frac{4}{3} \end{aligned} \right\}$$

4. A number of persons contribute a certain sum towards a charitable fund. If there had been 12 fewer, each would have contributed 13 as. 4p. more; if there had been 12 more, each would have contributed 8 as. less. Find the number of persons and the contribution of each.

5. Collect the co-efficient of x^3 from the product of $(3x^4 + x^3 - 5x^2 + 2x - 1)(5x^3 + 4x^2 - x + 3)$ without actual multiplication.

PAPER 10

1. The H.C.F. of two expressions is $a-1$, their L.C.M. is $a^5 + 4a^4 + 6a^3 + a^2 - 6a - 6$; one expression is $a^3 + a^2 - 2$, find the other.

2. Simplify $\left(\frac{1}{a} + \frac{\frac{1}{ab}}{\frac{1}{a} + \frac{1}{b}}\right) \left(\frac{1}{a} + \frac{\frac{1}{ab}}{\frac{1}{a} - \frac{1}{b}}\right) \div \frac{\frac{1}{a^2} + \frac{1}{b^2}}{\frac{1}{a^2} - \frac{1}{b^2}}$.

3. Solve the equations $\left. \begin{array}{l} x+y=9 \\ xy=20. \end{array} \right\}$

4. In a cyclic quadrilateral $ABCD$, $A=(2x+15)$ degrees, $B=(3x-25)$ degrees, $C=(3y-10)$ degrees and $D=(2y+10)$ degrees. Find x and y .

5. Factorise $a^3p^2 + 2a^3q^2 + b^3q^2 - 3a^3pq - 3b^3pq + 2b^3q^2$.

PART II

CHAPTER XI

LINEAR AND STATISTICAL GRAPHS

1. Length of Straight Line.

Example 1. The co-ordinates of P and Q are $(9, 3)$, $(-3, -2)$ respectively; find the length of PQ .

Draw $QNL \parallel x$ -axis and $PML \perp QNL$.

$$\begin{aligned} PQ^2 &= QL^2 + LP^2 \\ &= (12)^2 + (5)^2 \\ &= 144 + 25 = 169 \end{aligned}$$

$$\therefore PQ = \sqrt{169} = 13 \text{ units.}$$

As one unit = $\cdot 1''$,

$$\therefore PQ = 1.3''.$$

Otherwise.

$$\begin{aligned} \text{Since } QL &= QN + NL \\ &= -NQ + NL = NL - NQ \\ &= 9 - (-3). \end{aligned}$$

$$\begin{aligned} LP &= LM + MP = -ML + MP \\ &= MP - ML = 3 - (-2) \text{ and} \end{aligned}$$

$$PQ^2 = QL^2 + LP^2,$$

$$PQ^2 = \{9 - (-3)\}^2 + \{3 - (-2)\}^2$$

$$\therefore PQ = \sqrt{\{9 - (-3)\}^2 + \{3 - (-2)\}^2}.$$

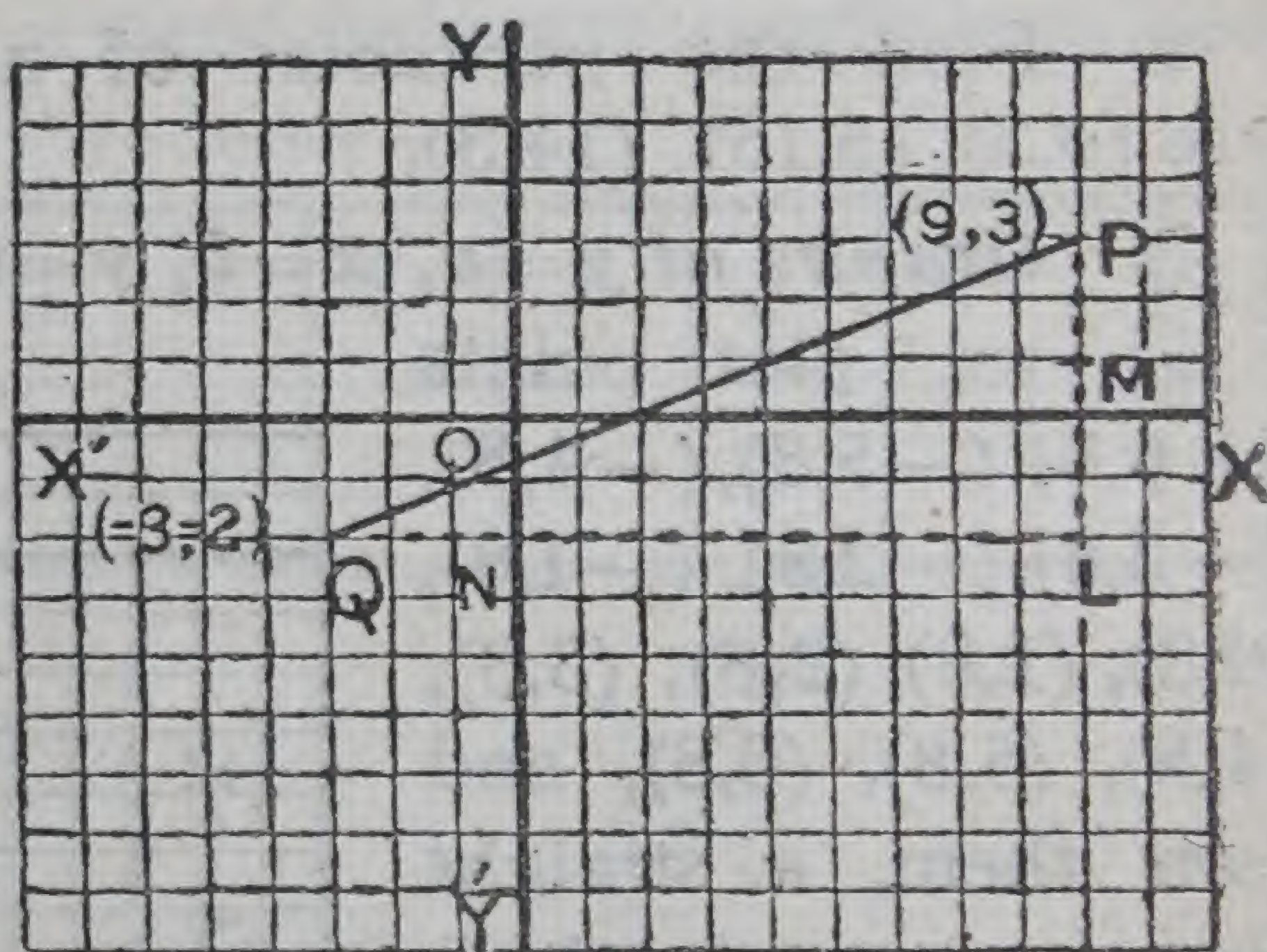
Generalising this process, we get the length of the line joining the points (x_1, y_1) and $(x_2, y_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$.

Cor. The distance of the point (x, y) from the origin $(0, 0) = \sqrt{x^2 + y^2}$.

EXERCISE 54.

1. Calculate the distance of each of the following points from the origin:

$$(6, 8), (10, 24), (8, 15), (7, 24).$$



2. Calculate the distance between the points of each pair :

(i) $(3,7), (6,3)$; (ii) $(4,8), (12,2)$; (iii) $(3,2), (15,7)$; (iv) $(2,3), (11,18)$.

3. Prove that the straight line joining the origin to the point $(8,18)$ subtends a right angle at the point $(13,5)$.

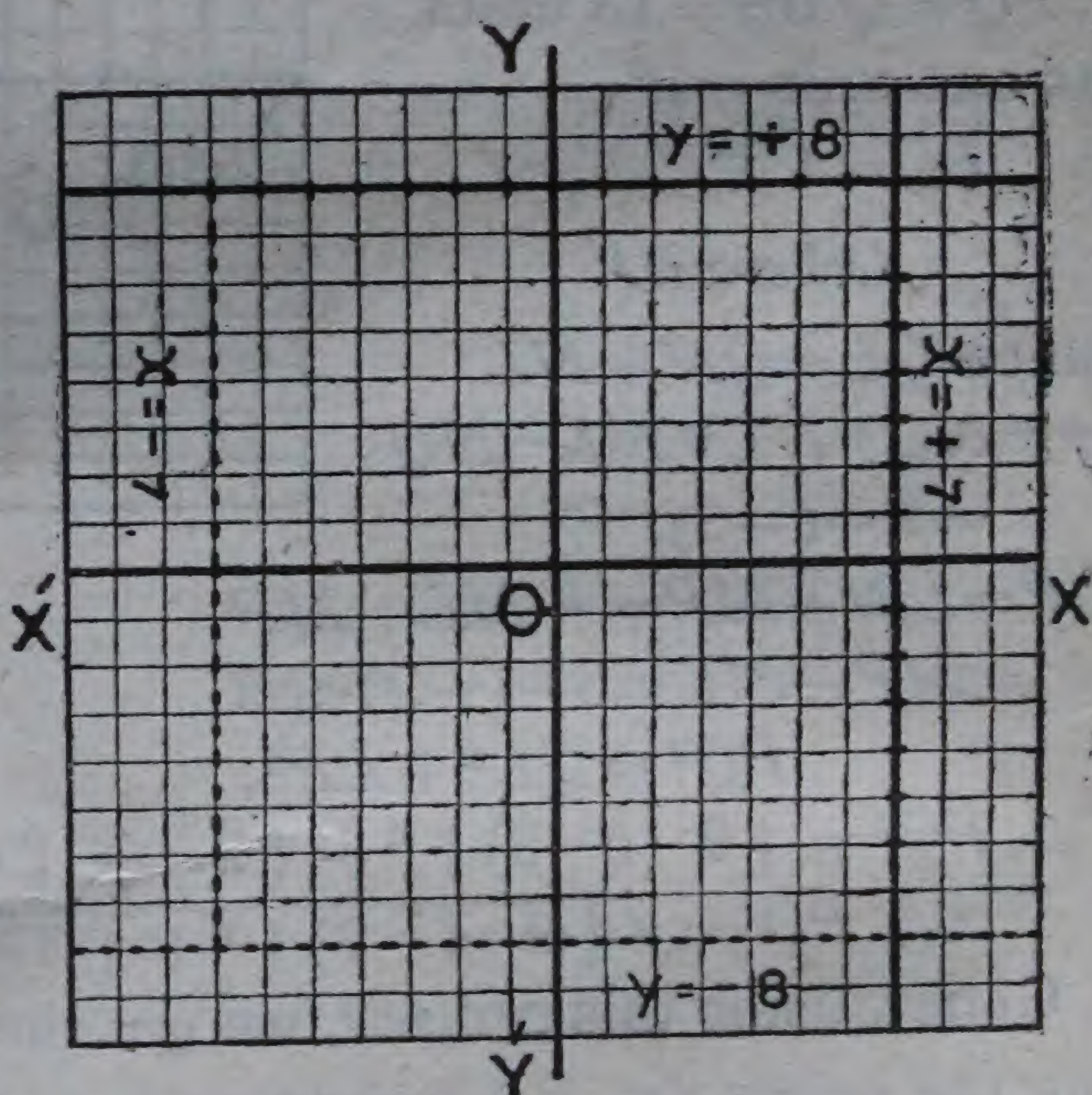
4. Prove that the triangle whose vertices are the points $(3,1), (11,7), (0,5)$, is a right-angled triangle.

5. Shew that the points $(-2,2), (6,2), (2,5)$ are the vertices of an isosceles triangle. Calculate the length of equal sides. Verify by measurement.

6. Find the perimeter of a triangle whose vertices are $(0,3), (5,15), (14,3)$.

7. **Graphs of $y=a, x=b, y=0$ and $x=0$.**

If we plot points $(-6,8), (-5,8), (-4,8), (-3,8), (-2,8), (-1,8), (0,8), (1,8), (2,8), (3,8), (4,8), (5,8), (6,8)$, and join them, a straight line is formed which is parallel to x -axis and is such that every point on it has its $y=+8$ and this is true for no other points. (See figure.)



The straight line is the **locus** of the points whose $y=+8$, whatever may be the x -co-ordinate, and the equation $y=+8$ represents this straight line.

Similarly, $y=-8$, represents a straight line parallel to x -axis and is such that every point on it has its $y=-8$.

Hence $y=a$, where a is constant, represents a straight line parallel to x -axis, and $y=0$ is the x -axis itself.

Again the straight line parallel to the y -axis and at the distance of $+7$ units is such that every point on it has $x = +7$ and this is true for no other points. (See figure.)

Hence $x = +7$ represents the straight line.

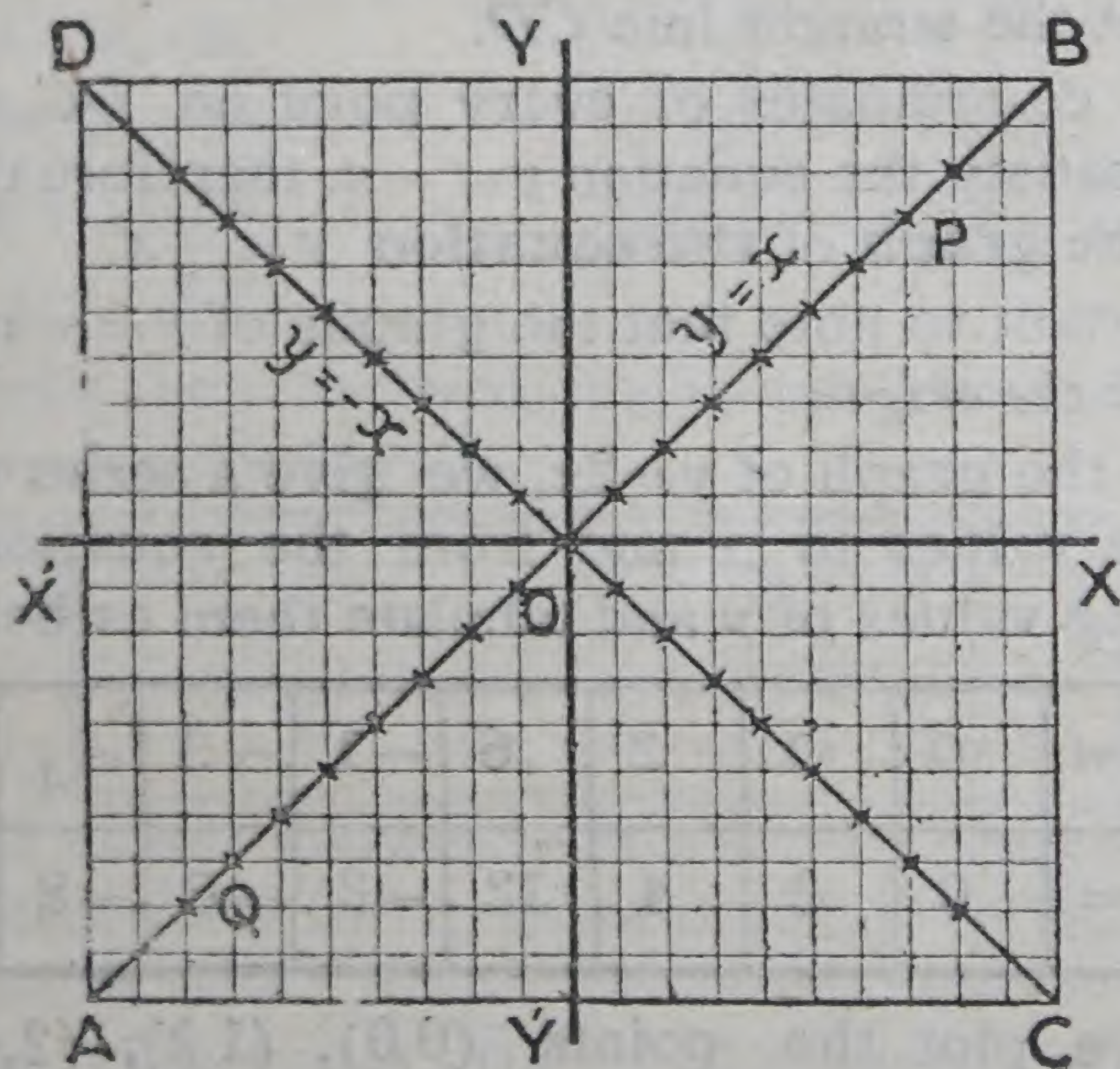
Similarly, $x = -7$ represents a straight line parallel to the y -axis and at the distance of -7 units from it.

Hence $x = b$, where b is constant, represents a straight line parallel to the y -axis, and $x = 0$ is the y -axis itself.

3. Graphs of (i) $y = x$, (ii) $y = -x$, (iii) $y = 2x$, (iv) $y = -2x$.

If we tabulate the values of x and the corresponding values of y from the equation $y = x$, we have

$x =$	-5	-4	-3	-2	-1	0	1	2	3	4	5
$y =$	-5	-4	-3	-2	-1	0	1	2	3	4	5



and if we plot the points $(-5, -5)$, $(-4, -4)$, $(-3, -3)$, $(-2, -2)$, $(-1, -1)$, $(0, 0)$, $(1, 1)$, $(2, 2)$, $(3, 3)$, $(4, 4)$, $(5, 5)$, etc., we find that all these points appear to lie on the straight line AB . (See figure.)

Now take *any* point P on the straight line AB . Its co-ordinates $(7,7)$ satisfy the equation $y=x$. Also take *any* point Q $(-8,-8)$ whose co-ordinates satisfy the equation $y=x$. We find that Q lies on the straight line AB .

Since the co-ordinates of every point on the straight line AB satisfy the equation $y=x$ and every solution of $y=x$ gives us the co-ordinates of a point on the straight line AB , the straight line AB is the **locus or the graph of the equation $y=x$** .

Similarly, by tabulating the values of x and y from the equation $y=-x$, as shown below:

$x=$	5	4	3	2	1	0	-1	-2	-3	-4	-5
$y=$	-5	-4	-3	-2	-1	0	1	2	3	4	5

and plotting the points $(5,-5), (4,-4), (3,-3), (2,-2), (1,-1), (0,0), (-1,1), (-2,2), (-3,3), (-4,4), (-5,5)$ and joining them, we get the straight line CD .

Since the co-ordinates of every point on DC and of no other point satisfy the equation $y=-x$, therefore the straight line CD is **the graph of the equation $y=-x$** .

It is important to note that the graphs of $y=x$ and $y=-x$ *pass through the origin*.

In finding the graph of $y=2x$, we give a series of positive and negative values to x and from the equation find the corresponding values of y and tabulate them as below:

$x=$	0	1	2	6	-1	-3	-4
$y=$	0	2	4	12	-2	-6	-8

Now if we plot the points $(0,0), (1,2), (2,4), (6,12), (-1,-2), (-3,-6), (-4,-8)$, we find that all these points lie on the straight line POQ .

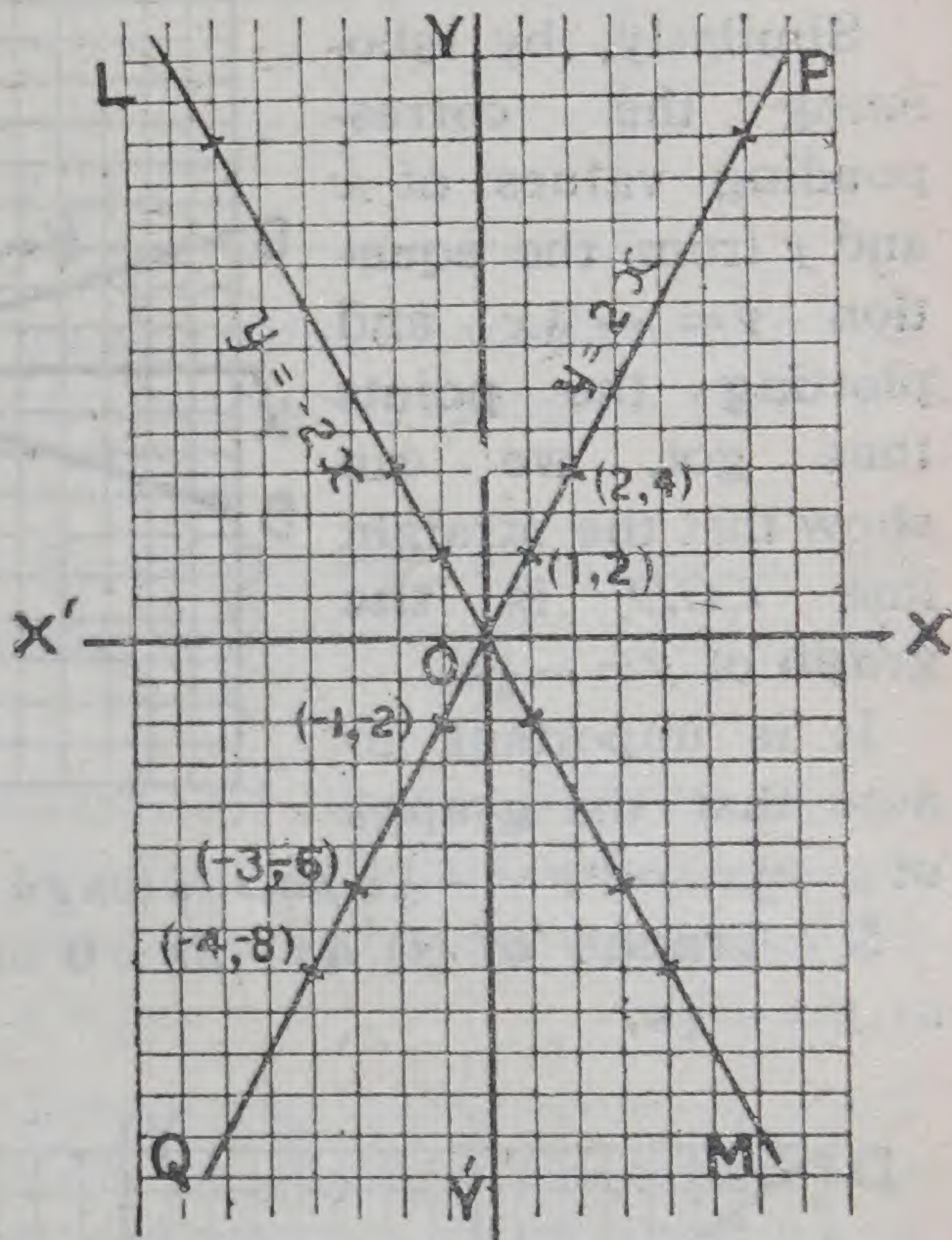
It can be easily verified that the co-ordinates of any point on the straight line POQ satisfy the equation $y=2x$.

and that the co-ordinates of no other point satisfy the equation.

Thus the straight line POQ is the graph of $y=2x$.

For the graph of $y=-2x$, we proceed exactly in the same way and find that the straight line LOM is the required graph.

It is important to note that the graphs of $y=2x$ and $y=-2x$ pass through the origin.



4. Graphs of

(i) $3y - x = 0$,
or $y = \frac{1}{3}x$;

(ii) $3y + x = 0$,
or $y = -\frac{1}{3}x$;

Tabulating the corresponding values of x and y from the equation $y = \frac{1}{3}x$, we have

$x =$	0	1	2	3	-2	-3
$y =$	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$-\frac{2}{3}$	-1

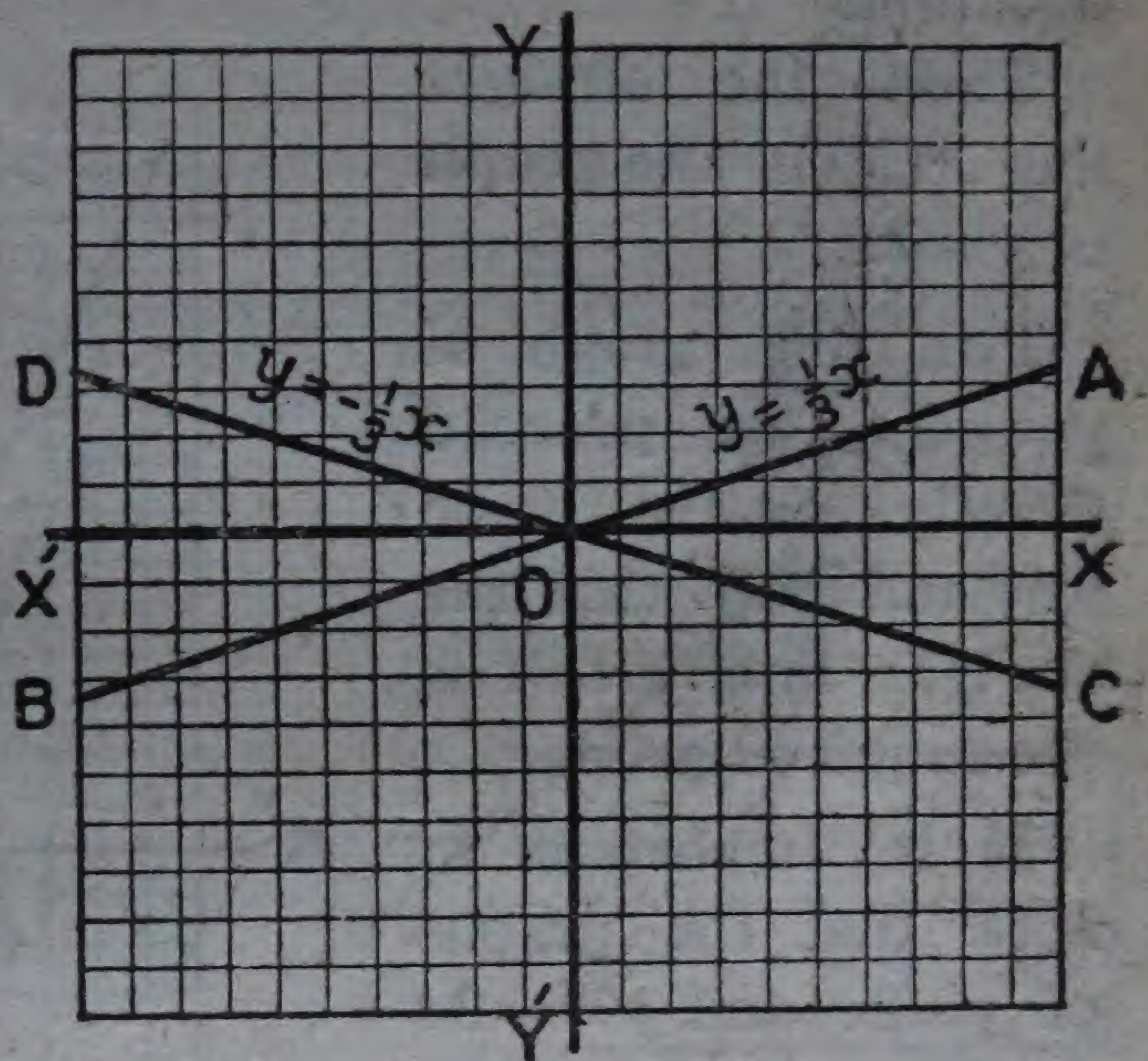
As these values involve fractions with denominator 3, it is therefore necessary for accurate drawing to take $\cdot 3''$ as the unit instead of $\cdot 1''$.

Now we plot points $(0,0)$, $(1, \frac{1}{3})$, etc., according to this scale and observe that all these points lie on the straight line AOB .

As it can be easily verified that the co-ordinates of any point on the straight line AOB satisfy the equation $y = \frac{1}{3}x$ and

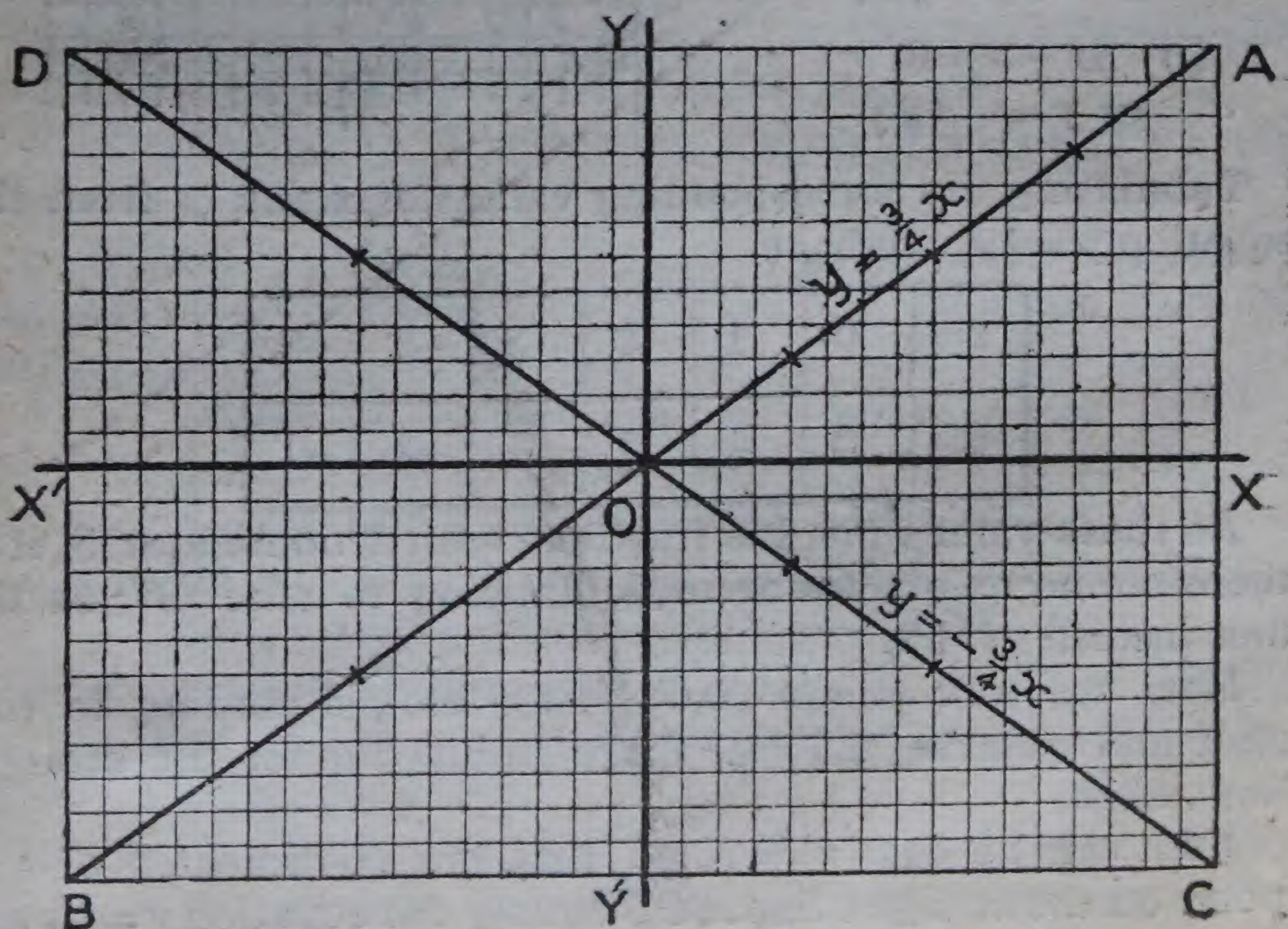
that the co-ordinates of no other point satisfy the equation the straight line AOB is the graph of $y = \frac{1}{3}x$.

Similarly, by tabulating the corresponding values of x and y from the equation $y = -\frac{1}{3}x$ and plotting the points thus got, we can show that the straight line COD is the graph of $y = -\frac{1}{3}x$.



It is important to note that the graphs of $y = \frac{1}{3}x$ and $y = -\frac{1}{3}x$ pass through the origin.

5. Graphs of (i) $4y - 3x = 0$ or $y = \frac{3}{4}x$; (ii) $4y + 3x = 0$, or $y = -\frac{3}{4}x$.



Tabulating the corresponding values of x and y from the equation $y = \frac{3}{4}x$, we have

$x =$	0	1	2	3	4	-2	-4
$y =$	0	$\frac{3}{4}$	$\frac{3}{2}$	$\frac{9}{4}$	3	$-\frac{3}{2}$	-3

As these values involve fractions with denominators 2 and 4, it is therefore necessary for accurate drawing to take $\cdot 4''$ as the unit instead of $\cdot 1''$.

Now we plot points $(0, 0)$, $(1, \frac{3}{4})$, etc., according to this scale and observe that all these points lie on the straight line AOB .

As it can be easily verified that the co-ordinates of any point on the straight line AOB and the co-ordinates of no other point satisfy the equation $y = \frac{3}{4}x$, therefore the straight line AOB is **the graph** of $y = \frac{3}{4}x$.

Similarly, by tabulating the corresponding values of x and y from the equation $y = -\frac{3}{4}x$ and plotting the points thus got, we can show that the straight line COD is **the graph** of $y = -\frac{3}{4}x$.

It is important to note that the graphs of $y = \frac{3}{4}x$ and $y = -\frac{3}{4}x$ pass through the origin.

The following points are worth noting :

(i) The equations $y - x = 0$, $y + x = 0$, $y - 2x = 0$, $y + 2x = 0$, $3y - x = 0$, $3y + x = 0$, $4y - 3x = 0$, $4y + 3x = 0$, $y = x$, $y = -x$, $y = 2x$, $y = -2x$, $y = \frac{1}{3}x$, $y = -\frac{1}{3}x$, $y = \frac{3}{4}x$, $y = -\frac{3}{4}x$, considered above, can be put into the form $ax + by = 0$.

Since $ax + by = 0$ can again be put into the form $y = \left(-\frac{a}{b}\right)x$ or $y = mx$, therefore each of the above equations can be put into the form $y = mx$.

(ii) In each case the graph is a straight line passing through the origin.

(iii) In each case m is the ratio between the ordinate and abscissa of a point on the graph and thus indicates the slope of the graph towards x -axis.

(iv) If m is positive, the graph lies in the first and third quadrants, if m is negative, the graph lies in the second and fourth quadrants.

EXERCISE 55.

1. If a point moves so that its ordinate is always -5 , find the path traced out by it.

2. If a point moves so that its abscissa is always -8 , find its locus.

3. Find the locus of a point which moves so that the ratio of its ordinate to its abscissa is $3 : 2$.

4. Find the locus of a point which moves so that the ratio of its ordinate to its abscissa is $-3 : 2$.

Draw the graphs of the following equations :

5. (i) $x = -7$, (ii) $x = 12$, (iii) $x = 0$, (iv) $x = -9$.
6. (i) $y = -6$, (ii) $y = 11$, (iii) $y = 0$, (iv) $y = -10$.
7. (i) $y = -\frac{1}{4}x$, (ii) $y = \frac{1}{5}x$, (iii) $y = -\frac{1}{6}x$.
8. (i) $y = \frac{3}{5}x$, (ii) $y = -\frac{3}{6}x$.
9. (i) $3x + 2y = 0$, (ii) $3x - 2y = 0$.
10. (i) $4x + 5y = 0$, (ii) $4x - 5y = 0$.
11. (i) $3x - 5y = 0$, (ii) $3x + 5y = 0$.
(iii) $3y - 5x = 0$, (iv) $3y + 5x = 0$.

6. Graphs of (i) $y = 2x$, (ii) $y = 2x + 6$, (iii) $y = 2x - 6$.

The corresponding values of x and y satisfying

(i) $y = 2x$ are

$x =$	0	1	2	3	4	-3	-4
$y =$	0	2	4	6	8	-6	-8

(ii) $y=2x+6$ are

$x=$	0	1	2	3	-2	-4
$y=$	6	8	10	12	2	-2

(iii) $y=2x-6$ are

$x=$	0	1	2	3	4	-2	-3
$y=$	-6	-4	-2	0	2	-10	-12

Now we plot points from table (i) and find that AOB is the graph of $y=2x$.

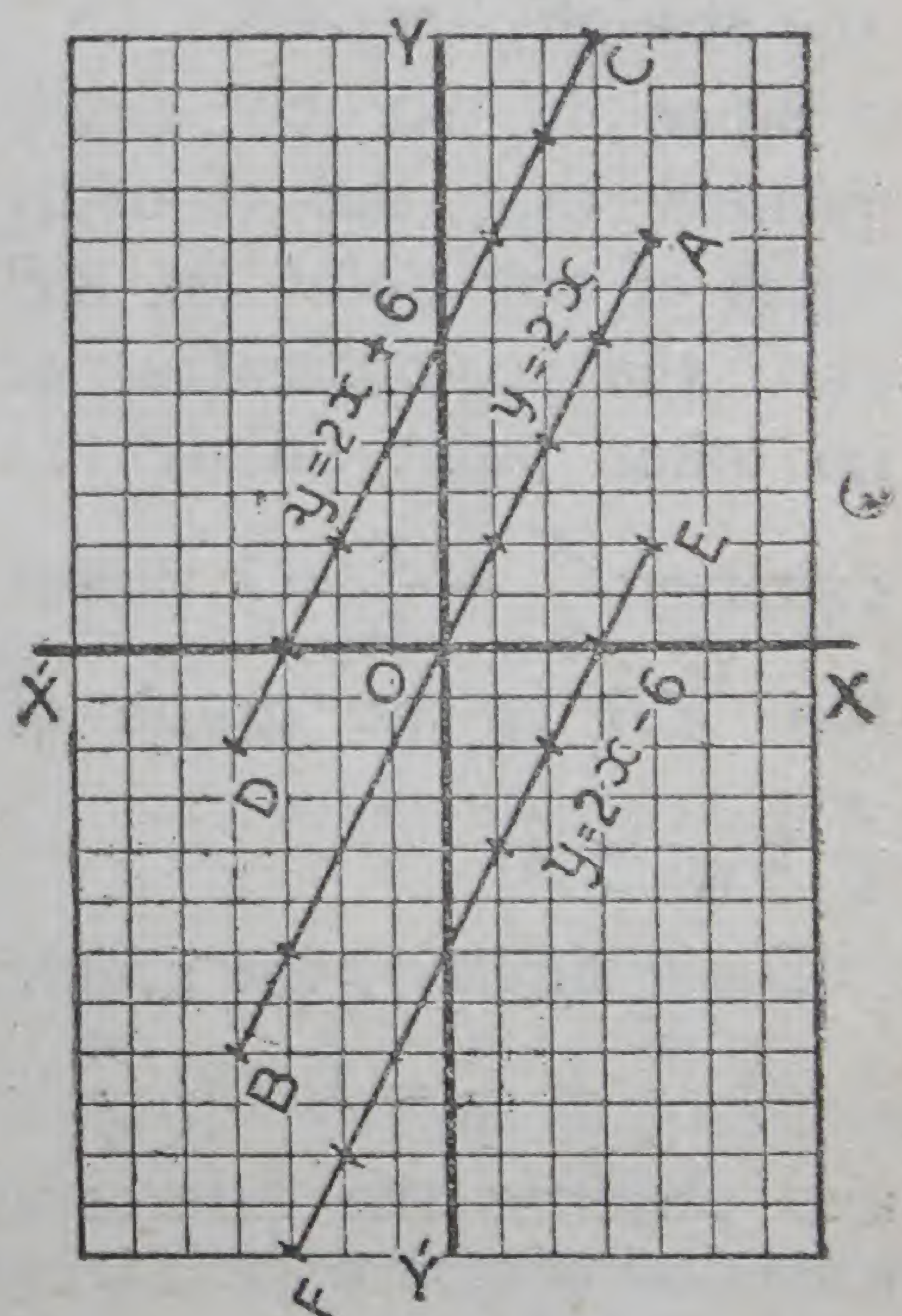
We plot points from table (ii) and find that CD is the graph of $y=2x+6$.

We plot points from table (iii) and find that EF is the graph of $y=2x-6$.

It will be observed that in each case the graph is a straight line and all these straight lines are *parallel*.

Moreover the intercept made by $y=2x+6$ on y -axis is $+6$, the intercept made by $y=2x$ on y -axis is 0 and the intercept made by $y=2x-6$ on y -axis is -6 , or the intercept made by the graph of an equation of the form $y=mx+c$ on y -axis is c .

It is important to note that any equation of the form $ax+by+c=0$ can be put into the form $y=\left(-\frac{a}{b}\right)x+\left(-\frac{c}{b}\right)$, or $y=mx+c$.



7. Graph of $\frac{x}{4} + \frac{y}{6} = 1$.

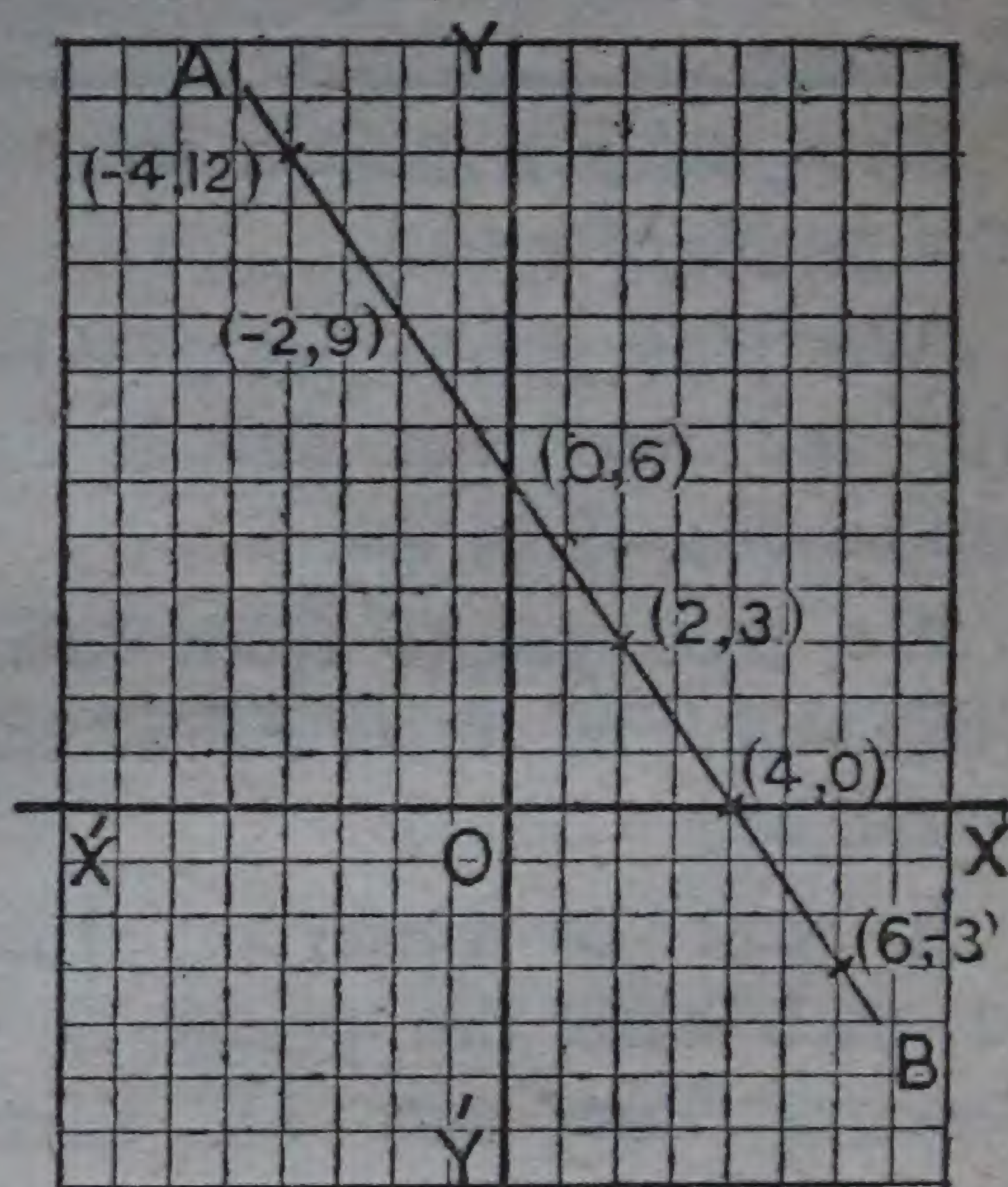
Tabulating the corresponding values of x and y from this equation, we have

$x =$	0	1	2	3	4	6	-2	-4
$y =$	6	$\frac{9}{2}$	3	$\frac{3}{2}$	0	-3	9	12

Plot the points $(0, 6)$, $(2, 3)$, $(4, 0)$, $(6, -3)$, $(-2, 9)$, $(-4, 12)$ with integral co-ordinates and join them.

All these points lie on the straight line AB .

Since it can be easily verified that the co-ordinates of any point on AB and the co-ordinates of no other point satisfy the equation $\frac{x}{4} + \frac{y}{6} = 1$, therefore AB is the graph of $\frac{x}{4} + \frac{y}{6} = 1$.



Draw similarly the graphs of the following equations :

(i) $\frac{x}{3} + \frac{y}{4} = 1$, (ii) $\frac{x}{4} + \frac{y}{5} = 1$, (iii) $\frac{x}{5} + \frac{y}{6} = 1$,

and observe the intercepts made by these lines on the axes and compare their length with the denominators of x and y in each case.

It is important to note that the straight line $\frac{x}{4} + \frac{y}{6} = 1$ cuts x -axis at $(4, 0)$ and y -axis at $(0, 6)$ or the intercept made by it on x -axis is 4 and is the same as the denominator of x in the equation, and the intercept made by it on y -axis is 6 and is the same as the denominator of y in the equation.

Similarly, the intercepts of $\frac{x}{3} + \frac{y}{4} = 1$ on the x -axis and y -axis are 3 and 4 respectively,

In *general*, if an equation is of the form $\frac{x}{a} + \frac{y}{b} = 1$, its graph is a straight line which makes with x -axis an intercept $= a$ and with y -axis an intercept $= b$.

From the above examples it is clear that the graph of an equation of the first degree in x and y , whatever be its form, is always a straight line.

It is for this reason that an equation of the first degree in x and y is generally called a **linear equation**.

Since a straight line is determined by any two points on it, therefore for drawing the graph of a linear equation, theoretically it is sufficient to plot two points on it, but for practical purposes and *accuracy* it is necessary to plot at least three points on it *sufficiently apart*. Integral values are more helpful than fractional ones. In most cases the values we get by putting $x=0$ and $y=0$ are very convenient.

Summary :

- (i) $(0,0)$ represents the origin.
- (ii) $y=0$ represents x -axis.
- (iii) $x=0$ represents y -axis.
- (iv) $x=a$ represents a straight line parallel to y -axis and at the distance of a from it.
- (v) $y=b$ represents a straight line parallel to x -axis and at the distance of b from it.
- (vi) $y=mx$ represents a straight line passing through the origin with m as its inclination towards x -axis.
- (vii) $y=mx+c$ represents a straight line with m as its inclination towards x -axis and c as its intercept on y -axis.
- (viii) Equations $y=mx+c_1$, $y=mx+c_2$, etc., which differ in the absolute term only, represent straight lines parallel to $y=mx$.
- (ix) $\frac{x}{a} + \frac{y}{b} = 1$ represents a straight line having intercepts a and b on x -axis and y -axis respectively.

8. Graph of a Linear Function of x .

An expression which contains a variable x and which has a definite value for every value of x is called a **function of x** .

Thus $3x^2 + 2x - 5$, $3x + 5$ are functions of x .

Such expressions as $6x$, $7x + 3$, $\frac{3x-5}{4}$ which contain no power of x higher than the first are called **linear functions of x** .

Suppose we have to draw the graph of the linear function $\frac{2x-6}{3}$.

Here we put $y = \frac{2x-6}{3}$ and by giving different values to x find the corresponding values of y .

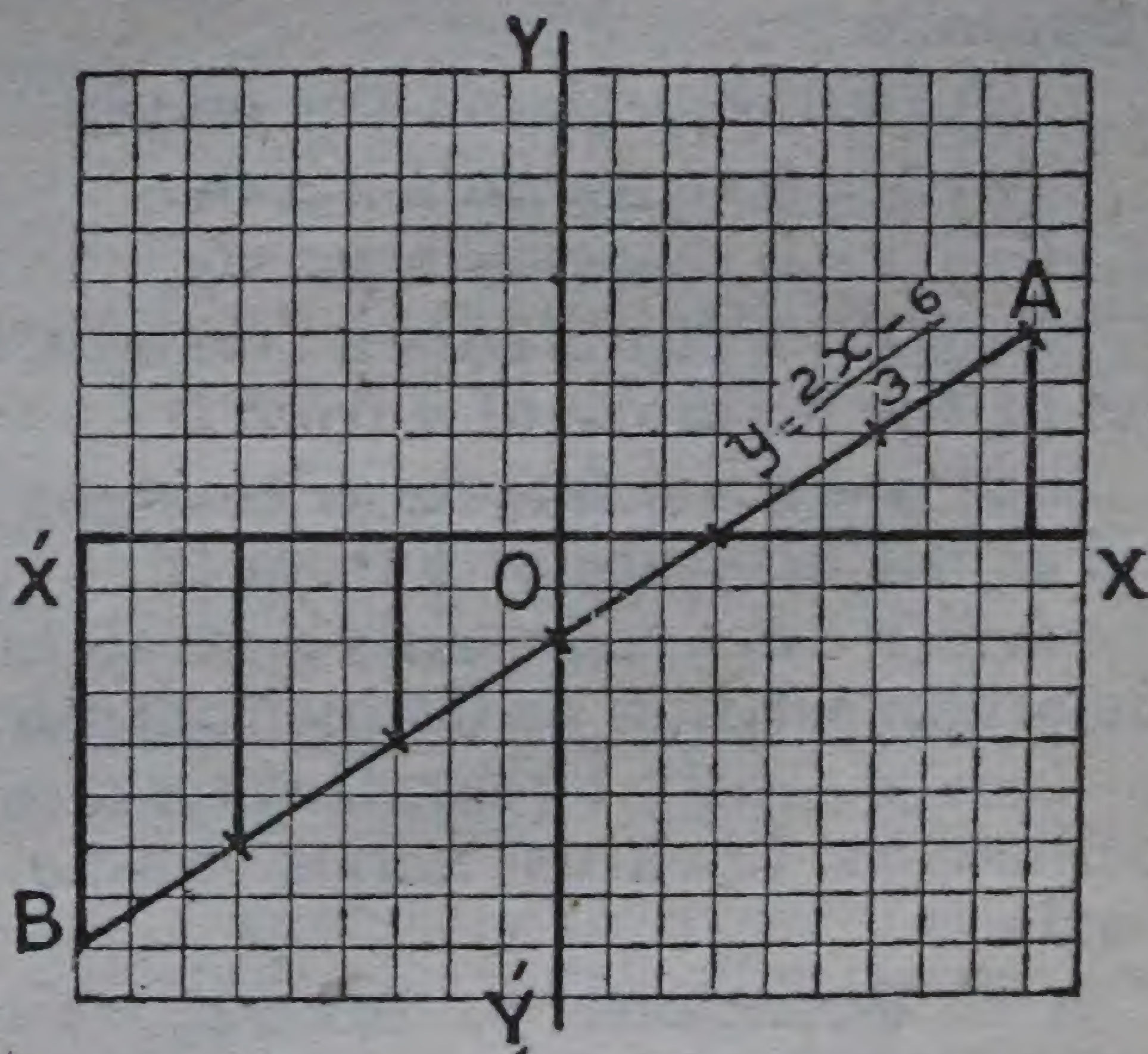
As this is an equation of the first degree, its graph must be a straight line.

Now we choose 3 convenient points on this straight line as given below :

$x =$	0	3	6
$y =$	-2	0	2

Plot points $(0, -2)$, $(3, 0)$, $(6, 2)$ and join them. The straight line AB is the required graph.

From this graph we can read the value of the function of x , for different values of x , e.g., when



$$(i) \ x=9, \quad \frac{2x-6}{3}=4; \quad (ii) \ x=-3, \quad \frac{2x-6}{3}=-4;$$

$$(iii) \ x=-6, \quad \frac{2x-6}{3}=-6; \quad (iv) \ x=-9, \quad \frac{2x-6}{3}=-8.$$

9. Equation of a straight line passing through two points.

Let the given points be (9, 4) and (6, 2) and the required equation of the straight line be $y = mx + c$... (i)

As (9, 4) lies on (i), $\therefore 4 = 9m + c$... (ii)

and as (6, 2) „ „ „ $\therefore 2 = 6m + c$... (iii)

From (ii) and (iii) $m = \frac{2}{3}$ and $c = -2$,

\therefore the required equation is $y = \frac{2}{3}x - 2$ or $2x - 3y = 6$.

EXERCISE 56.

[Choose a suitable scale wherever necessary, use accurate squared paper and a pencil with a sharp end.]

1. In the same diagram draw the graphs of:

- (i) $y = 4x + 5$, (ii) $y = -4x + 5$, (iii) $y = 4x - 5$,
(iv) $y = -4x - 5$.

2. In the same diagram draw the graphs of:

- (i) $3y = x + 6$, (ii) $3y + x = 6$, (iii) $x - 3y = 6$,
(iv) $x + 3y + 6 = 0$.

3. In the same diagram draw the graphs of:

- (i) $y = 5x$, (ii) $y = 5x + 4$, (iii) $y = 5x - 4$.

4. In the same diagram draw the graphs of:

- (i) $y + 2x = 0$, (ii) $y + 2x = 5$, (iii) $y + 2x + 5 = 0$.

Reduce each of the following equations to the form ;

- (i) $y = mx + c$, (ii) $\frac{x}{a} + \frac{y}{b} = 1$, (iii) $ax + by + c = 0$,

5. $x = 2y + 5$. 6. $x + \frac{1}{2}y = 4$. 7. $3x - 4y = 10$.

8. $2y = \frac{1}{4}(x - 3)$. 9. $\frac{x}{3} - \frac{y}{4} + 1 = 0$. 10. $\frac{4}{5}x + \frac{5}{4}y = 1$.

Reduce each of the following equations to the form $y = mx + c$:

11. $y = 3$. 12. $x = 5$. 13. $y = 0$. 14. $x = 0$.

Draw the graphs of the following equations:

- | | | |
|------------------------------------|-----------------------------------|------------------------------------|
| 15. $x+3=0$. | 16. $y+7=0$. | 17. $3y+5x=0$. |
| 18. $5y-3x=0$. | 19. $5y+3x=0$. | 20. $8y+6x=5$. |
| 21. $8x-6y=5$. | 22. $x-4y=9$. | 23. $x-4y+9=0$. |
| 24. $4x=2y+6$. | 25. $x=11y-4$. | 26. $x+3y=\frac{9}{2}$. |
| 27. $\frac{x}{5}+\frac{y}{3}=1$. | 28. $\frac{x}{5}-\frac{y}{3}=1$. | 29. $\frac{x}{7}+\frac{y}{9}=1$. |
| 30. $\frac{x}{7}-\frac{y}{9}=1$. | 31. $\frac{x-y}{3}=\frac{3}{4}$. | 32. $\frac{y}{3}=\frac{7x-6}{5}$. |
| 33. $\frac{y}{3}=4\frac{1}{2}-x$. | | |

34. Draw the graphs of the following equations in the same diagram and find the area included by them :

- (i) $y-3=0$, (ii) $x-2=0$, (iii) $4x+5y=60$.

35. Draw the graph of the equation $3x+4y=8$. In this graph read the value of x when y is -10 . [Punjab, 1918.]

36. Plot the graph of $y=\frac{3x+13}{2}$ and from the graph read off the value of y when $x=3$. What are the intercepts on the axes of x and y ?

Show that the points $(-1, 5)$ and $(1, 8)$ lie on the graph.

Draw the graphs of the following functions of x :

- | | | |
|------------------------|------------------------|------------------------|
| 37. $(4-3x)$. | 38. $8(3-x)$. | 39. $\frac{1-x}{2}$. |
| 40. $\frac{3x-4}{5}$. | 41. $\frac{5x-3}{6}$. | 42. $\frac{7x+3}{6}$. |

Find the equations of the straight lines passing through the following pairs of points:

- | | |
|-------------------------|--------------------------|
| 43. $(4, 5), (6, 8)$. | 44. $(-3, 5), (-5, 3)$. |
| 45. $(0, -4), (5, 0)$. | 46. $(0, 0), (-6, 8)$. |

47. For what value of m and c will the points $(3, 5)$, $(-3, 8)$ lie on the straight line $y=mx+c$?

48. Plot points $(3, -3)$ and $(-3, 3)$ and write down the equation of the straight line joining them. [Punjab, 1918.]

49. Plot points $(12, 4)$, $(-3, -5)$ and find if the straight line joining them passes through the point $(8, 2)$. [Punjab, 1919.]

50. Plot points $A(5, 7)$, $B(7, 10)$, $C(2, 2)$ and $D(-1, -4)$ and find the equations of the straight lines AB and CD . Find the point of their intersection. [Bombay, 1912.]

10. Graphic Solution of Simultaneous Equations.

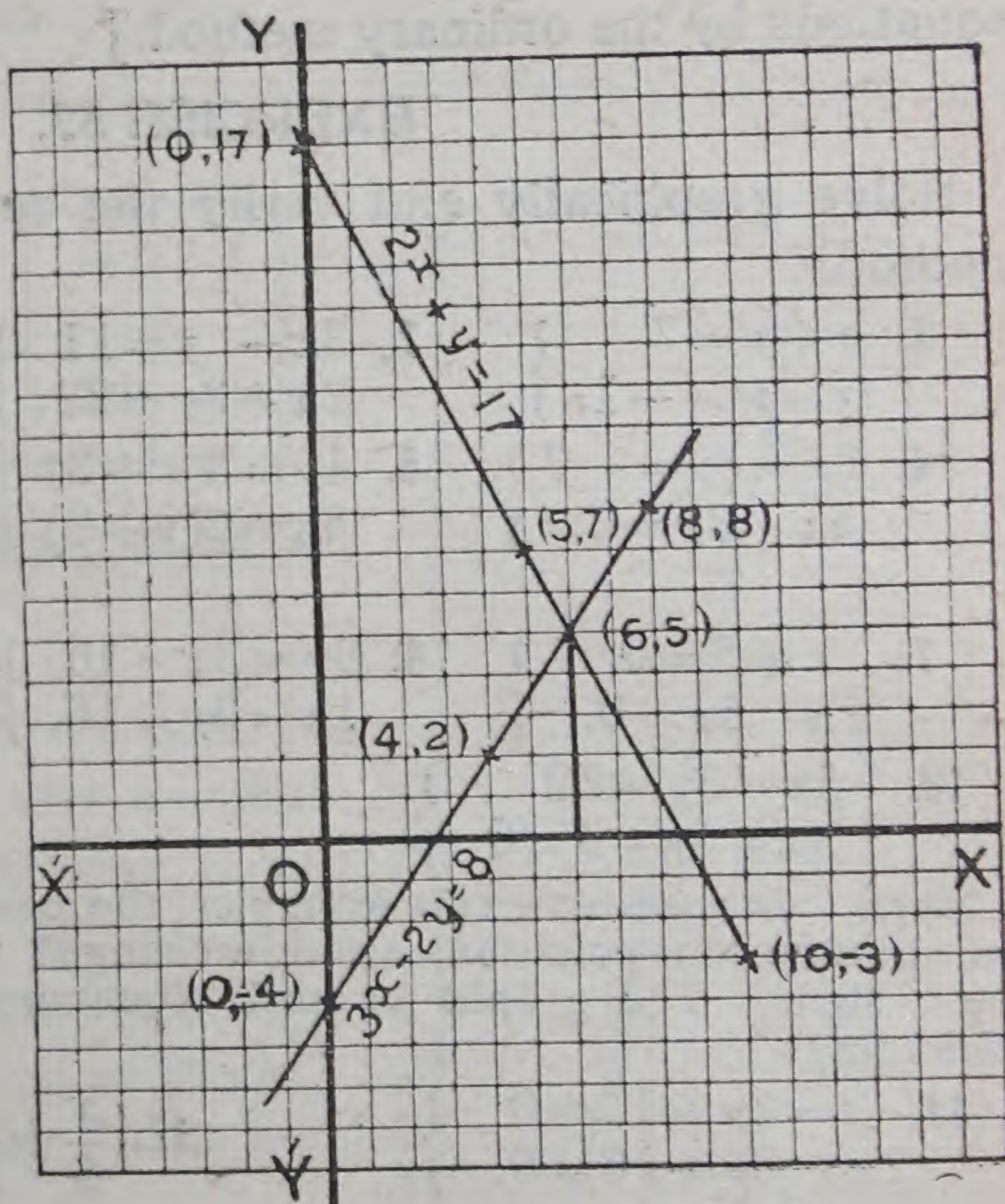
Example. Solve graphically the equations:

$$2x + y = 17 \quad \dots \quad (i)$$

$$3x - 2y = 8 \quad \dots \quad (ii)$$

As both the equations are of the first degree in x and y , their graphs must be straight lines.

Now plot 3 points on each straight line sufficiently apart, as given below.



From equation $2x + y = 17$, we have

$x =$	0	5	10
$y =$	17	7	-3

From equation $3x - 2y = 8$, we have

$x =$	0	4	8
$y =$	-4	2	8

We join the points $(0, 17)$, $(5, 7)$ and $(10, -3)$ for the graph of $2x + y = 17$ and the points $(0, 4)$, $(4, 2)$ and $(8, 8)$ for the graph of $3x - 2y = 8$, and write down the corresponding equations on the graphs for the sake of distinction.

As these graphs intersect at the point $(6, 5)$, the co-ordinates $(6, 5)$ satisfy both the equations.

Hence $x = 6$ and $y = 5$ is the solution of the given equations.

[The students may verify the result by solving the equations by the ordinary method.]

EXERCISE 57.

Solve graphically and verify the result by the ordinary method:

- | | | | | | |
|--------------------|---|--------------------------|---|--|---|
| 1. $x + y = 7$ | } | 2. $3x - y = 11$ | } | 3. $2x + 3y = 17$ | } |
| $x - y = -1$ | | $2x + 3y = 22$ | | $3x + 2y = 18$ | |
| 4. $2x - 3y = -7$ | } | 5. $4y = 3(x + 2)$ | } | 6. $2x = 3y$ | } |
| $3x - 2y = 2$ | | $5x = 2(y + 2)$ | | $\frac{x}{2} + \frac{y}{6} = 1\frac{5}{6}$ | |
| 7. $x + 5 = 0$ | } | 8. $2y = 3x - 16$ | } | 9. $3x + 4y = 24$ | } |
| $2x - 5y = 0$ | | $\frac{1}{2}x = 3y + 16$ | | $4x - 3y = 7$ | |
| 10. $5x - 3y = 29$ | } | | | | |
| $3x + 5y + 3 = 0$ | | | | | |

NOTE. In the above two examples, the co-efficients of x and y in the simultaneous equations are interchanged with a single change in their signs. The graphs of such equations are perpendicular to each other.

- | | | | |
|-----------------------|---|--------------------------------------|---|
| 11. $x - 2y + 11 = 0$ | } | 12. $\frac{x}{5} + \frac{y}{10} = 1$ | } |
| $2x - 3y + 18 = 0$ | | $3x - 2y = 1$ | |

[Punjab, 1924]

Show that the following straight lines are concurrent, also find the co-ordinates of the common points:

13. $x + y = 11$, $2x - y - 4 = 0$, $x + 2y = 17$.

14. $2x + y = 13$, $3x - 2y - 2 = 0$, $x + 11 = 3y$.

15. $4x - y = 5$, $5x + 2y = 16$, $x + 3y = 11$.

16. Find the co-ordinates of the vertices and also the lengths of the sides of the triangle formed by the straight lines,

$x - y = 0$, $x + 5 = 0$, $x + y = 4$.

11. Application of Graphs—Statistical Graphs.

Example. If sugar is selling @ 4 seers a rupee, we can have 8 seers of sugar for Rs. 2, 16 seers for Rs. 4, 20 seers for Rs. 5, and so on.

Here the ratio between the quantity of sugar in seers and the price in rupees = 4 : 1.

If the price in rupees be represented by x and the quantity in seers by y , then $y = 4x$.

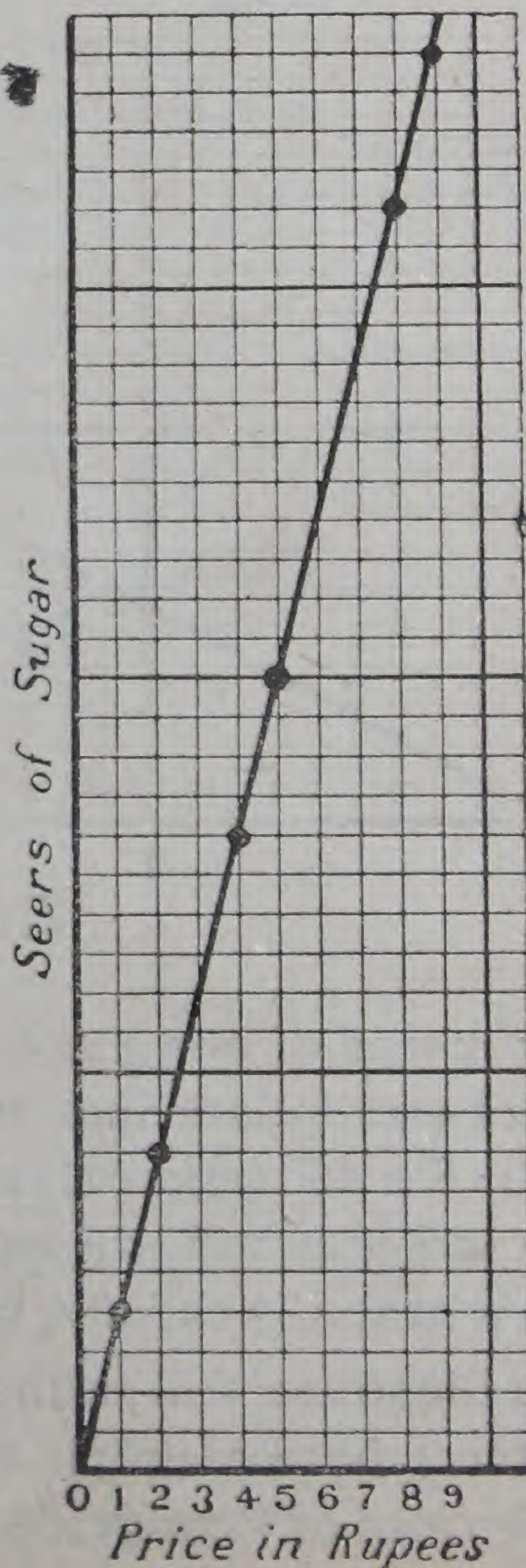
Tabulating the values of x and y from this equation we have

$x =$	0	1	2	4	5
$y =$	0	4	8	16	20

and plotting these points, we have the annexed graph.

Now we can use this graph to *read off* the price of any number of srs. of sugar and *vice versa*, e.g., the price of 14 srs. is Rs. 3-8 as., for when $y = 14$, $x = 3\frac{1}{2}$, and Rs. 4 $\frac{1}{2}$ is the price of 18 srs., for when $y = 18$, $x = 4\frac{1}{2}$.

Thus the graph can be used as a *ready reckoner*. The limit up to which it can be used as such, depends on the size of the graph. By increasing our unit and using larger paper, we can obtain a wider range of prices and quantities and also greater accuracy.



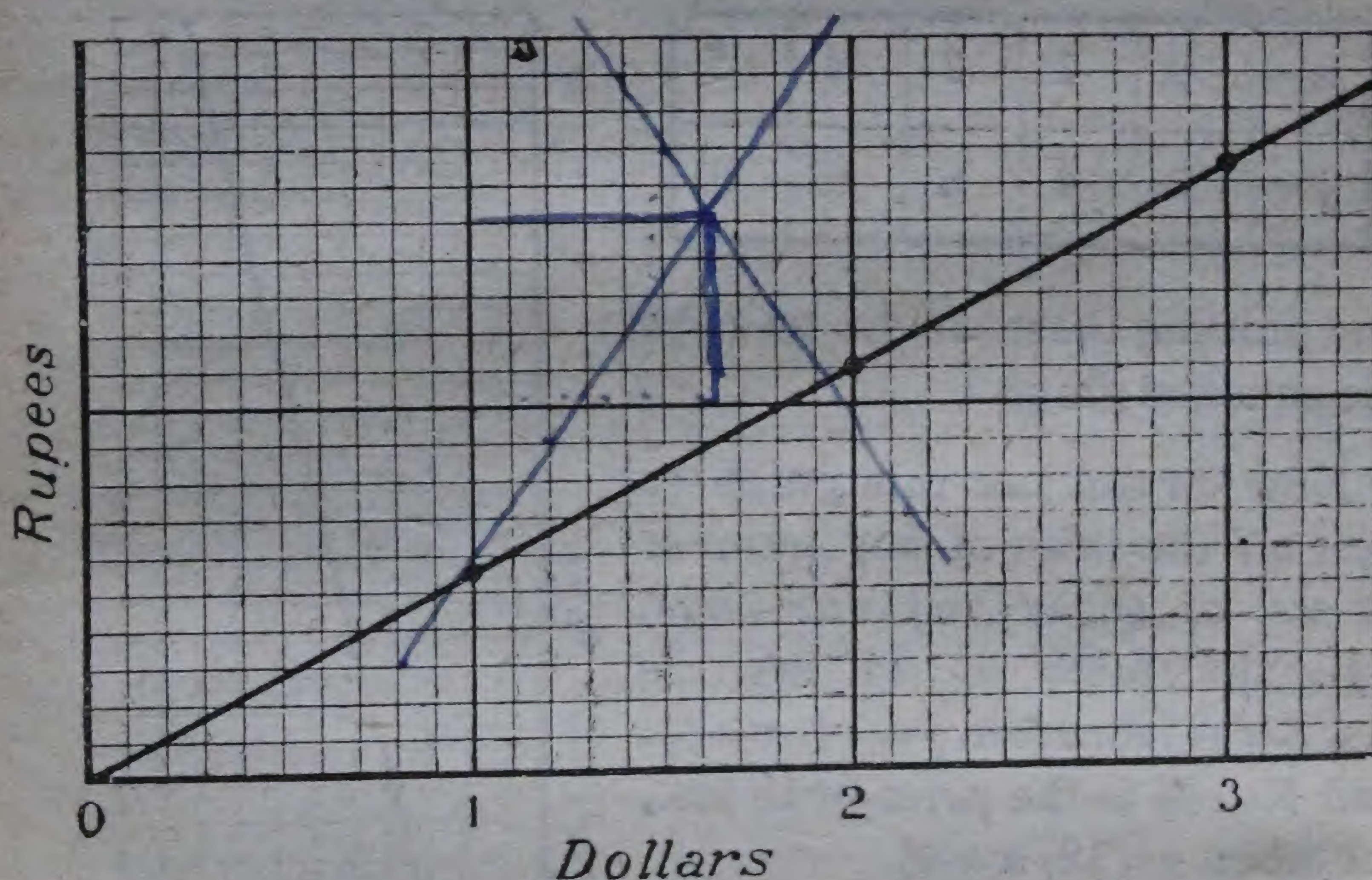
12. Conversion Graph. \$1 = Rs. 2.75 (approx.), therefore if y represents the number of rupees and x the number of corresponding dollars, we have

$$y = 2.75x$$

Tabulating the values of x and y from this equation, we have

$x =$	0	1	2	3
$y =$	0	2.75	5.5	8.25

Taking 1" for \$1 along the x -axis and .2" for Re. 1 along the y -axis, we have the following graph:



By means of our graph we can now convert rupees into dollars and dollars into rupees with some accuracy. Thus \$ 1.5 = Rs. 4.2 (approx.) and Rs. 6 = \$ 2.2 (approx). By drawing a larger graph, we can increase its range of application. Such a graph is called a **conversion graph**.

To illustrate still further the use of conversion graphs we construct here a graph representing the relation between the degrees of Centigrade and Fahrenheit scales.

Since $32^{\circ}\text{F} = 0^{\circ}\text{C}$ and 100°C corresponds to 212°F and there are $(212^{\circ} - 32^{\circ})$ or 180°F for 100°C , $\therefore \frac{9}{5}^{\circ}\text{F} = 1^{\circ}\text{C}$.

Hence the relation between the number of degrees in the two scales is given by the equation

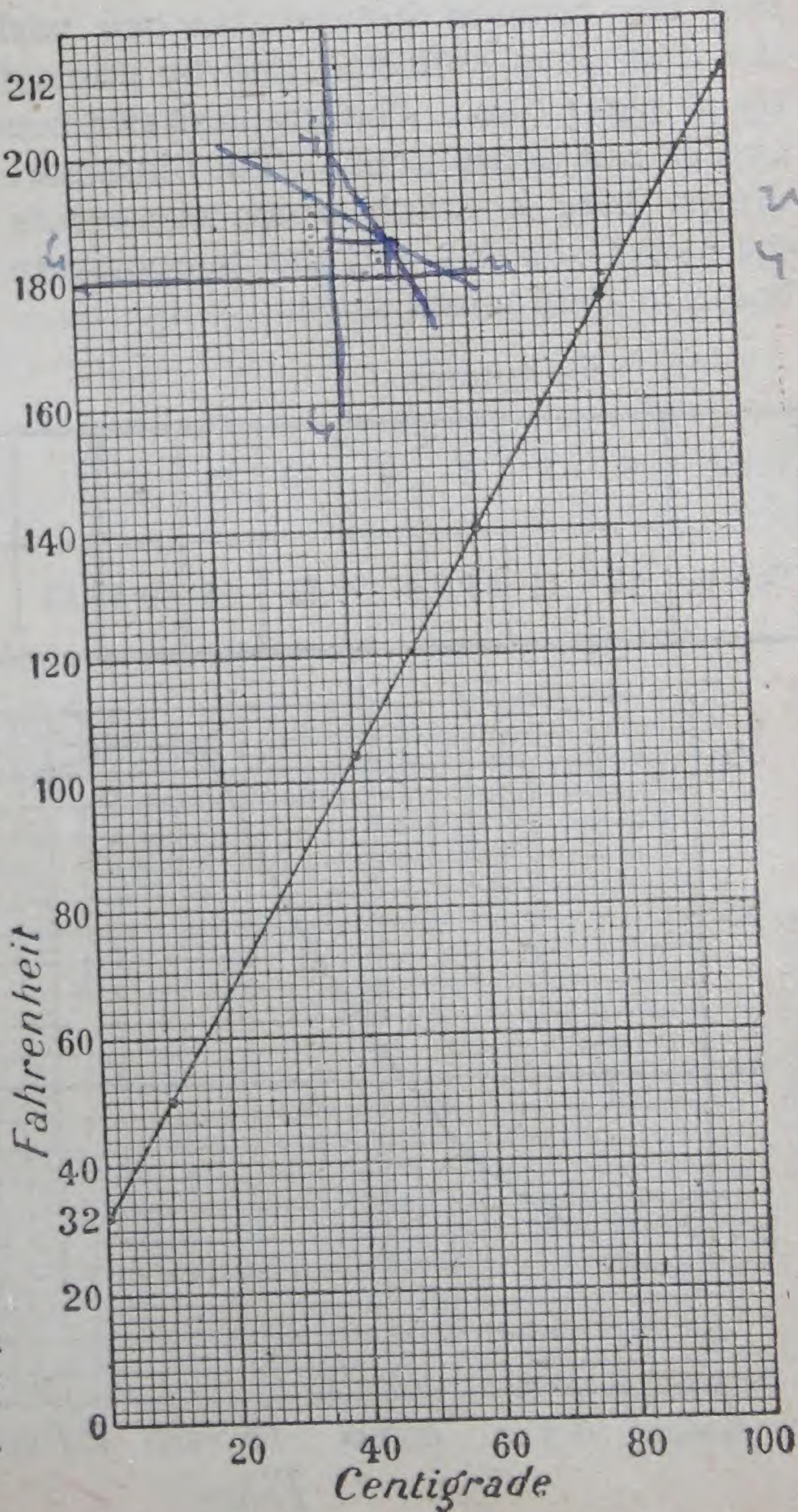
$$y = \frac{9}{5}x + 32,$$

where x stands for the number of degrees in the Centigrade scale and y for the number of degrees in the Fahrenheit scale.

Tabulating the values of x and y from the above equation, we have

$x =$	0	10	40	60	80
$y =$	32	50	104	140	176

Taking 1 cm. to represent 20°C along x -axis and 20°F along y -axis and plotting the points, we get the annexed graph. From this graph read the value of 50°C in Fahrenheit scale and that of 120°F in Centigrade scale.



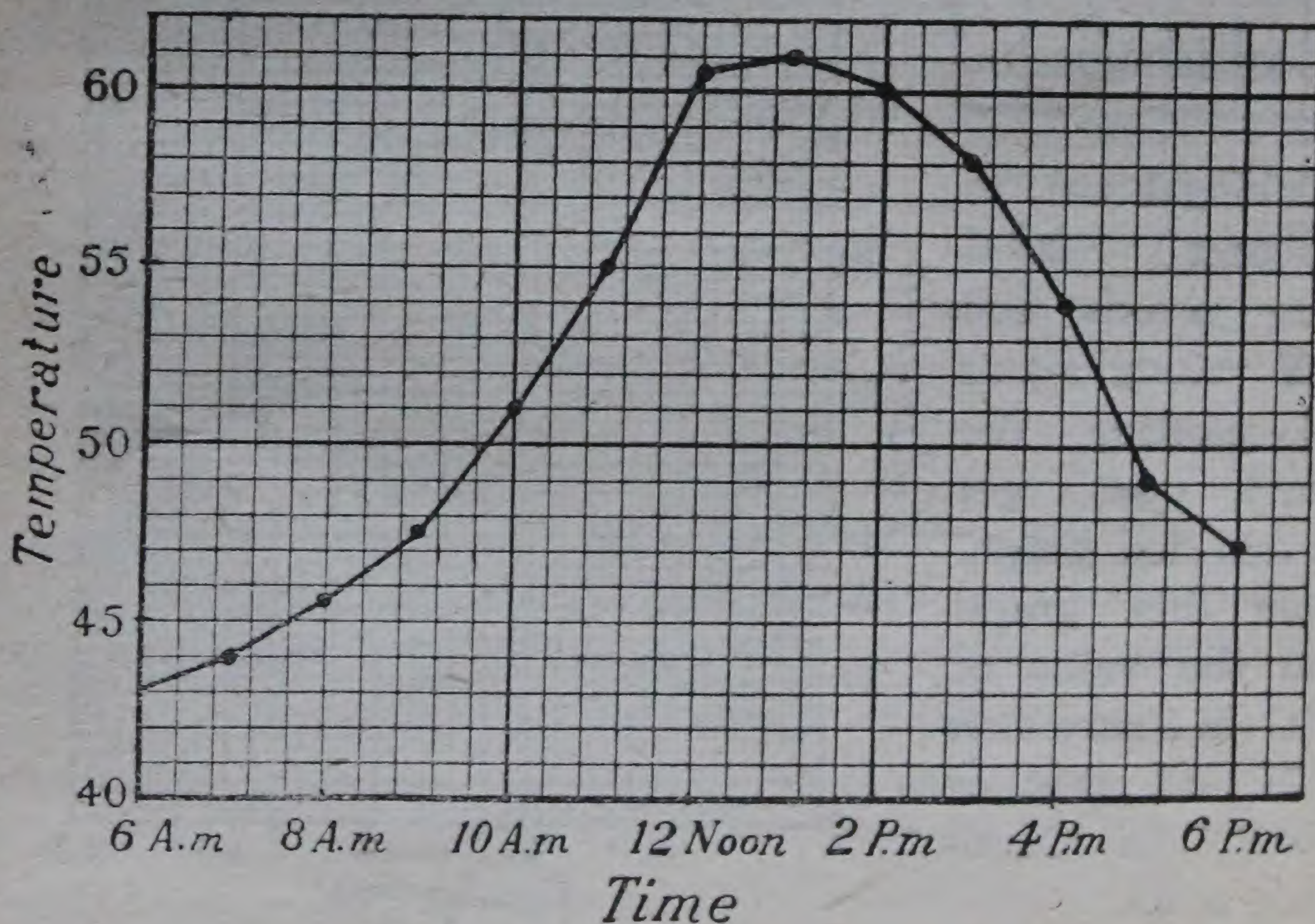
13. Discontinuous and Continuous Graphs.

Sometimes the corresponding values of two *variables* cannot be determined from a given relation or an equation but are obtained by actual observation or experiments, *e.g.*, the temperature at different intervals of time throughout a given period. In such cases we draw the graph representing the relation between the two variables with the help of the data thus collected and by joining the successive points by *straight lines*. The graph drawn consists of *separate lines* and is not continuous. To illustrate this point we have drawn here a graph which represents the relation between the hours of a day and the temperature of a place at such hours in Fahrenheit scale.

Forenoon

Afternoon

Time	6	7	8	9	10	11	12	1	2	3	4	5	6
Temp.	43	44	45.5	47.5	51	55	60.5	61	60	58	54	49	47



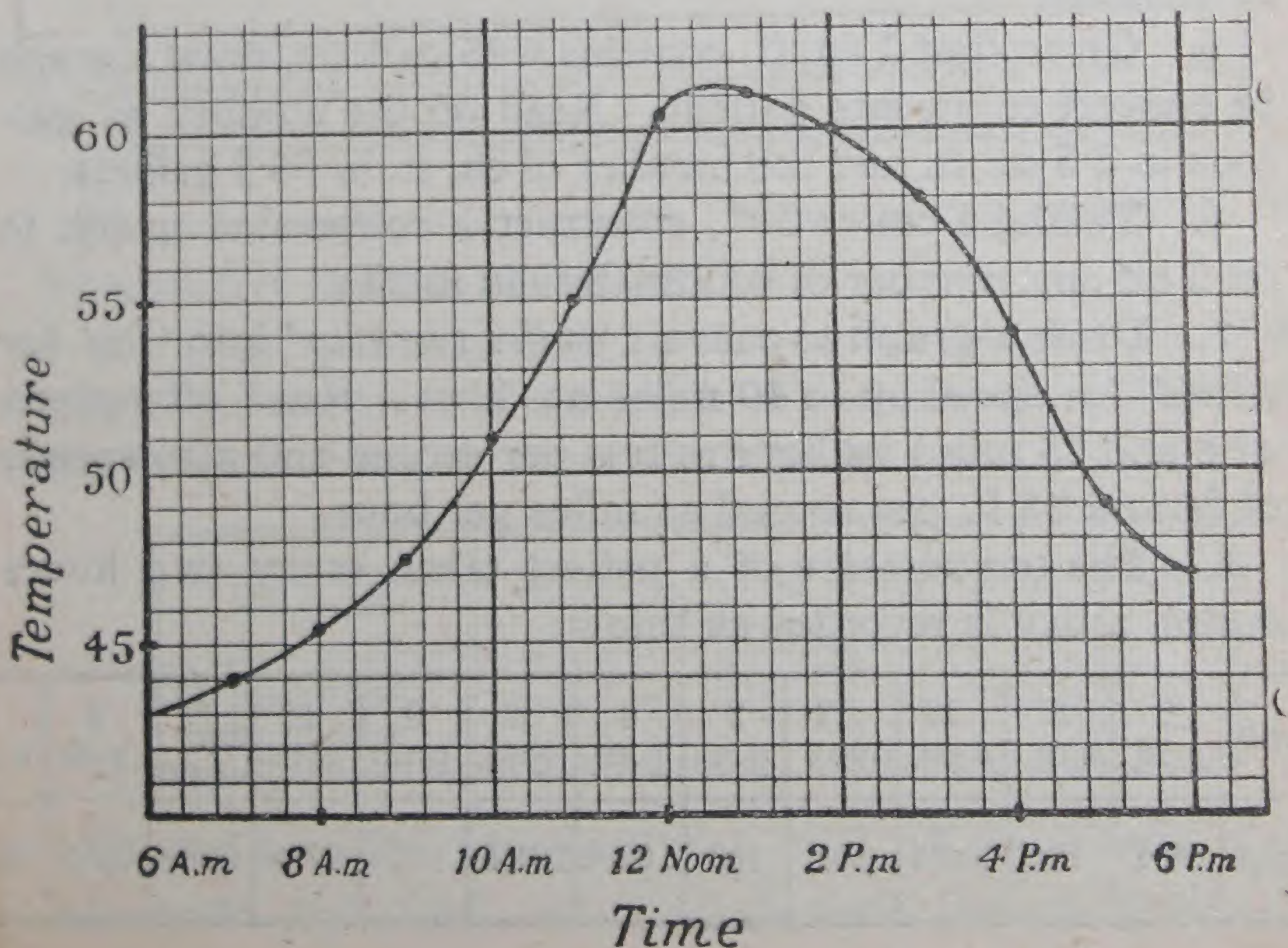
In this case we have taken 6 A.M. and 40° as the *origin*, 25" for 1 hour along the x -axis and .1" for 1° along the y -axis. The successive points are joined by separate st. lines and we get a discontinuous graph.

Now we can read the approximate temperature at any particular time :

e.g., at 10-30 A.M. temp. is 53° (approx.)
and at 2-30 P.M. temp. is 58.8° (approx.)

At 1 P.M. the temperature appears to be the *maximum*, and between 12 noon and 2 P.M. the temperature is fairly steady. Between 10 A.M. and 12 noon there is a rapid increase, and between 3 P.M. and 5 P.M. there is a rapid fall.

In all experimental work, since the observations are made at intervals, the graphs made from the data thus collected would be discontinuous. Really in majority of cases the variation is *not so abrupt but gradual and continuous*, therefore the proper graph in such cases should be continuous, as shown below :



This graph gives us better representation of the variation of temperature during the day and also gives us more accurate reading, *e.g.*, the maximum temperature is not exactly at 1 P.M. but between 12 noon and 1 P.M.

EXERCISE 58.

1. If rice cost 1 anna 6 pies per seer, construct a graph to read off the price of any number of seers. Frame an equation connecting the price of rice in annas and the number of seers.

2. If 40 articles cost Re. 1 4 as., construct a graph to read off the price of articles correct up to 6 pies.

3. Taking $\pounds 1 = \text{Rs. } 13\frac{1}{2}$, construct a conversion graph to read off the number of shillings for a given number of rupees.

4. Taking 1 kilometre = .62 mile (approx.), draw a graph showing the number of miles in any number of kilometres up to 100. Read off approximately the number of kilometres in 45 miles.

5. Given that 1 cu. ft. contains 6.25 gallons, draw a graph to convert cu. ft. into gallons. Read off the number of gallons in 8.5 cu. ft. and the number of cu. ft. in 66.5 gallons.

6. Taking 1 cm. = .39", construct a conversion graph to read off any number of centimetres in inches.

7. Draw a graph to convert '*miles per hour*' into '*feet per second*' for speed up to 40 miles an hour. Read off speeds of 8 and 25 miles an hour as feet per second and also speeds of 24 and 35 ft. per second as miles per hour.

8. The temperature of a patient taken every two hours during a day is recorded as below :

Time	6 a.m.	8 a.m.	10 a.m.	12 noon	2 p.m.	4 p.m.	6 p.m.	8 p.m.	10 p.m.	12 mid- night	2 a.m.	4 a.m.
Temp.	99.2	100	100.4	101	103	102.4	101.6	100	99.8	99.6	99.4	99.2

Draw a continuous graph and read off the temperature at 3 P.M. and the time when the temperature was approximately 100.5° .

9. The average monthly temperature of a town is as follows :

Month	Jan.	Feb.	Mar.	Ap.	May	Jun.	July	Aug.	Sep.	Oct.	Nov.	Dec.
Temp.	63°	72°	88°	95°	102°	106°	104°	100°	96°	90°	75°	65°

Draw a continuous graph and read off approximately the hottest fortnight of the year.

10. The length of the longest day in different latitudes is given below in hours :

Latitude	0°	10°	20°	30°	40°	50°	60°	70°	80°	90°
Length of the longest day	12	12.6	13.3	14	14.8	16.1	18.5	24	24	24

Draw a continuous graph. Estimate the length of the longest day in latitudes 16° , 38° , 65°

11. The following table gives us the heights of places above the sea-level in thousands of feet and the corresponding heights of the barometer in inches :

Height above sea-level	0	5	10	15	20	25
Height of barometer	30.1	25	20.6	17.1	14.3	11

Draw a continuous graph and read off the height of the barometer at 3,000 ft. and 8,000 ft. above the sea-level.

12. Draw the graph illustrating the annual premium for Life Assurance for Rs. 1000 according to the ages at the time of the first Payment from the following table :

Age in years	22	24	26	28	30	32	34	36
Premium in Rs. & as.	25	26-2	27-6	28-12	30-6	32-2	34	36-2

Read off the premium for starting at the ages (i) 25, (ii) 33.

13. The temperature of water in a boiler is 20°C . It is continually heated for half an hour and its temperature is recorded at intervals of 5 minutes and is tabulated below :

Intervals in minutes	5	10	15	20	25	30
Temperature	24°	28.6°	32.8°	36.6°	40°	43°

Show graphically the relation between time and temperature of the water and read off the temperature after 23 minutes.

14. The formula $s=16t^2$ gives us the relation between the space (s) covered by a body falling freely under gravity in time (t). If in the following table s be given in feet and t in seconds,

t	1	2	3	4	5	6
s	16	64	144	256	400	576

draw a continuous graph for the relation between the distance and time, and read off

(i) the height of a tower from the top of which a stone takes 4.5 seconds to fall,

(ii) the depth of water-level of a well which a stone takes 2.25 seconds to reach.

Verify the results by calculation.

15. The average height of boys and girls at different ages is given below in centimetres in two different tables :

Table for boys

Age in years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ht, in centimetres	68	80	89	93	95	100	107	116	122	124	126	128	132	140	149	156	160	164

Table for girls

Age in years	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Ht, in centimetres	68	79	84	88	95	105	111	116	117	120	124	128	140	148	150	154	156	158

Draw *carefully* both these graphs on the same paper with the same axes. Represent one by dotted lines and the other by a continuous line.

Note the following points :

- (1) The age up to which both the curves are concurrent.
- (2) The ages at which the two curves intersect.
- (3) The ages when the springs of boys and girls are almost parallel.
- (4) The ages when the plateaus of boys and girls are almost parallel.
- (5) The ages when the boys are in springs and the girls in plateaus.

Note that the *plateaus* are the periods when the system can be more energetic and the *springs* are the periods when the system is more liable to fatigue.

Taking this as a law, point out periods :

- (i) when the boys can be made to do more work,
- (ii) when the girls can be made to do more work,
- (iii) when the boys should have a lighter course,
- (iv) when the girls should have a lighter course,
- (v) at what ages should we hold the public examinations for boys and girls ?

16. The average weights of boys and girls are given in lbs. up to the age of 15 years, in the following tables :

Table for boys

Age	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Weight	22	36	38	42	45	50	53	58	63	69	71	74	80	95	106

Table for girls

Age	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Weight	22	32	35	40	44	47	50	54	58	64	71	80	90	109	109

Draw both these graphs on the same paper with the same axes and compare the curves.

CHAPTER XII

RATIO

1. If A and B be two quantities of the *same kind* which when measured with one and the same unit are represented by the numbers a and b , the **ratio** of A to B is the quotient of a by b .

This ratio is written as a/b , $a \div b$ or $a : b$ and is read as 'the ratio of a to b .'

Thus, for example, if A and B be lengths of $3\frac{1}{4}$ ft. and $5\frac{1}{2}$ ft. respectively, the ratio of A to B is $3\frac{1}{4}/5\frac{1}{2}$ which when simplified $= \frac{13}{22}$ or $13 : 22$.

Similarly, the ratio of Rs. $3\frac{1}{4}$ to Rs. $5\frac{1}{2}$ is $13 : 22$.

In the first example the quantities compared are *two lengths* and in the second *two sums of money*; the ratio in both cases is the same.

Thus, a ratio is simply an abstract number and is independent of the concrete units employed in the quantities compared.

In the ratio $a : b$, a and b are its **terms**. The first term a is called the **antecedent** and the second term b is called the **consequent**.

The ratio $b : a$ is known as the **inverse ratio** of $a : b$.

As a ratio is represented by a vulgar fraction, *its value remains unaltered if its terms are multiplied or divided by the same number*; for instance, $\frac{a}{b} = \frac{ma}{mb}$ and $\frac{na}{nb} = \frac{a}{b}$.

A ratio is said to be a ratio of *greater inequality*, of *equality* or of *less inequality*, according as the antecedent is greater than, equal to, or less than the consequent.

Thus $4 : 3$ or $\frac{4}{3}$ is a ratio of greater inequality

$3 : 3$ or $\frac{3}{3}$ " " " equality,

$3 : 4$ or $\frac{3}{4}$ " " " less inequality.

Two or more ratios are said to be **compounded**, when they are multiplied together. Thus the ratio compounded of $a : b$ and $c : d$ is $ac : bd$.

When the ratio $a : b$ is compounded with itself, the resulting ratio $a^2 : b^2$ is called the **duplicate** ratio of $a : b$.

Similarly, $a^3 : b^3$ is called the **triplicate** ratio of $a : b$. When $a : b$ is the original ratio, $\sqrt{a} : \sqrt{b}$ is called the **sub-duplicate** ratio and $\sqrt[3]{a} : \sqrt[3]{b}$ is called the **sub-triplicate** ratio of $a : b$.

2. Theorem. *A ratio of greater inequality is diminished and a ratio of less inequality is increased, by adding the same positive quantity to both the terms.*

Let $\frac{a}{b}$ be a ratio, a and b being positive.

Let x , a positive quantity, be added to both its terms.

The new ratio thus formed $= \frac{a+x}{b+x}$.

$\frac{a}{b} > = < \frac{a+x}{b+x}$ according as $\left(\frac{a}{b} - \frac{a+x}{b+x}\right)$ is positive, zero or negative.

$$\text{Now } \frac{a}{b} - \frac{a+x}{b+x} = \frac{ax - bx}{b(b+x)} = \frac{x(a-b)}{b(b+x)}.$$

Since x , b , $(b+x)$ are all positive, the above is positive, zero or negative according as $a-b$ is positive, zero or negative; or according as $a > = < b$.

$$\therefore \frac{a}{b} > = < \frac{a+x}{b+x} \text{ according as } a > = < b.$$

It is left to the student to prove the corresponding theorem:

A ratio of greater inequality is increased and a ratio of less inequality is diminished, by subtracting the same positive quantity from both the terms.

Example 1. Find the ratio compounded of the three ratios $4a : 5b$, $b^2 : 2ac$ and $3c^2 : 2ab$.

$$\text{The required ratio} = \frac{4a \times b^2 \times 3c^2}{5b \times 2ac \times 2ab} = 3c : 5a.$$

Example 2. If $4+x:5+x$ be the duplicate ratio of $2:3$, find the value of x .

$$\begin{aligned}\frac{4+x}{5+x} &= \frac{2 \times 2}{3 \times 3} = \frac{4}{9} \\ \therefore 36+9x &= 20+4x \\ \therefore 5x &= -16 \\ \therefore x &= -3\frac{1}{5}.\end{aligned}$$

Example 3. What must be added to the terms of the ratio $4:7$ to make it equal to $5:6$?

Let x be the required number.

$$\begin{aligned}\text{Then by the question } \frac{4+x}{7+x} &= \frac{5}{6} \\ \text{or } 6(4+x) &= 5(7+x) \\ \therefore 24+6x &= 35+5x \\ \therefore x &= 11.\end{aligned}$$

Example 4. If $x:y=4:7$, find the value of $\frac{2x+y}{x+3y}$

Dividing the numerator and the denominator by y

$$\text{we have } \frac{2x+y}{x+3y} = \frac{2\left(\frac{x}{y}\right) + 1}{\frac{x}{y} + 3}$$

Substituting the value of $\frac{x}{y}$, we get

$$\begin{aligned}\frac{2x+y}{x+3y} &= \frac{2 \times \frac{4}{7} + 1}{\frac{4}{7} + 3} \\ &= \frac{1\frac{8}{7}}{3\frac{4}{7}} = \frac{15}{25} = \frac{3}{5}.\end{aligned}$$

Example 5. If $3x+4y=5x-2y$, find the ratio $x:y$.

Since $3x+4y=5x-2y$

$$\therefore 2x=6y$$

$$\text{or } x=3y$$

$$\therefore \frac{x}{y} = 3 = \frac{3}{1}$$

$$\therefore x:y = 3:1.$$

EXERCISE 59.

[Do the first five questions *mentally*.]

1. Write down the ratio compounded of the ratios of :
 - (i) $3 : 4, 5 : 6, 2 : 3$;
 - (ii) $a : b, b : c, c : d$;
 - (iii) $(x+1) : (x-1), (x^2 - x + 1) : (x^2 + x + 1)$.
2. Write down the duplicate ratio of :
 - (i) $3 : 7$. (ii) $ab : cd$, (iii) $(a+x) : (a-x)$,
 - (iv) $(x+1) : (x-1)$.
3. Write down the triplicate ratio of :
 - (i) $4 : 5$, (ii) $mn : pq$, (iii) $(a-b) : (a+b)$, (iv) $x^2 : 2x$.
4. Write down the sub-duplicate ratio of :
 - (i) $16 : 25$, (ii) $x^6 : y^8$, (iii) $x^2 + 4ax + 4a^2 : x^2 - 4ax + 4a^2$.
5. Write down the sub-triplicate ratio of :
 - (i) $64 : 27$, (ii) $x^9 : y^6$, (iii) $125a^6b^{12} : 343x^3y^6$.
6. Compare the ratios :
 - (i) $2 : 3$ and $5 : 6$, (ii) $14 : 11$ and $8 : 5$.
7. If $4x + 3 : 7x - 1$ be the duplicate ratio of $3 : 4$, find the value of x .
8. If $m : n$ be the duplicate ratio of $2m - x : n - 2x$, shew that $x^2 = mn$.
9. If $(a+x) : (a-x)$ be the duplicate ratio of $(a+b) : (a-b)$, then $(x-b) : (a-x) = b(a+b) : a(a-b)$.
10. If $p : q$ be the triplicate ratio of $p+x : q+x$, shew that $x^3 - 3pqx - pq(p+q) = 0$.
11. What must be added to the terms of $5 : 9$ to make it equal to $7 : 8$?
12. What must be added to the terms of $a : b$ so that the resulting ratio may be in the duplicate ratio of $a : b$?
13. What must be added to the terms of $a : b$ so that the resulting ratio may be in the triplicate ratio of $a : b$?

14. Two numbers are in the ratio of 7 : 8 and if four be added to each, they will be in the ratio of 8 : 9. Find the numbers.

[*Hint.* Let $7x$ and $8x$ be the required numbers.]

15. Two numbers are in the ratio of 3 : 4 and if five be subtracted from each, they will be in the ratio of 2 : 3. Find the numbers.

16. Find two numbers in the ratio of 9 : 7 whose difference is 11.

17. Two numbers are in the ratio of 3 : 5 and the sum of their squares is 1666; find them.

18. If x be added to the antecedent of the ratio $a : b$, what quantity must be added to the consequent so that the value of the ratio may remain unaltered?

19. If $x : y = 5 : 6$, find the value of $\frac{5x-3y}{6x+3y}$.

20. If $x : y = 3 : 2$, find the value of $\frac{4x^2 - xy + 2y^2}{3x^2 + xy - y^2}$.

21. If $\frac{7x-4y}{3x+y} = \frac{5}{13}$, find the value of $\frac{x}{y}$.

22. If $\frac{2x^2-3y^2}{x^2+y^2} = \frac{2}{41}$, find $\frac{x}{y}$.

23. If $(8x-2y)^2 = (3x+4y)^2$, find $x : y$.

24. Which is greater, $\frac{y+x}{y-x}$ or $\frac{y^2+x^2}{y^2-x^2}$, when x and y are both positive and $y > x$?

25. Which is greater, $\frac{x^3+y^3}{x^2+y^2}$ or $\frac{x^2+y^2}{x+y}$ when x and y are both positive?

26. Find the least integer which must be added to each term of the ratio 3 : 5 so that the resulting ratio may be greater than 2 : 3.

27. Find two numbers whose difference is 9 and whose sub-duplicate ratio is 1 : 2.

28. The ages of two persons are in the ratio of 5 : 4; 9 years ago they were in the ratio 4 : 3. Find their ages.

29. The bases of two triangles are in the ratio of 3 : 4 and their heights in the ratio of 6 : 5. Find the ratio of their areas.

30. The ratio of the radii of two circles is $p : q$ and the ratio of their areas is $p - x : q - x$. Express x in terms of p and q .

*31. At a height of h feet we can see to a distance of $\sqrt{\frac{3h}{2}}$ miles. An aeroplane ascends from 2,000 ft. to 5,000 ft.; in what ratio does the pilot increase his range of vision?

*32. The soldiers in two armies when they met in a battle were in the ratio of 10 : 3. Their respective losses were as 20 : 3 and the survivors as 40 : 13; if the number of survivors in the larger army be 24,000, find the original number of soldiers in each army.

Particular Cases of Ratio

*3. Meaning of $\frac{0}{n}$, $\frac{m}{\infty}$, $\frac{m}{0}$, $\frac{0}{0}$.

(i) The fraction $\frac{m}{n}$ stands for the share of each person when a quantity m is divided among n persons.

Similarly, $\frac{0}{n}$ stands for the share of each person when 0 is divided among n persons. As in this case the quantity to be divided is zero, the share of each must be zero.

Or $\frac{0}{n} = 0$ (i)

Cor. When $n = 1$, $\frac{0}{1} = 0$.

(ii) If the numerator of a fraction is constant (say m), its value decreases as its denominator increases, e.g., $\frac{m}{3} < \frac{m}{2}$, $\frac{m}{4} < \frac{m}{3}$, $\frac{m}{5} < \frac{m}{4}$... and when the denominator is very